ADJOINT VECTOR FIELDS AND DIFFERENTIAL OPERATORS ON REPRESENTATION SPACES

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ABSTRACT. Let G be a semisimple algebraic group with Lie algebra $\mathfrak{g}$. In 1979, J. Dixmier proved that any vector field annihilating all G-invariant polynomials on $\mathfrak{g}$ lies in the $\Bbbk[\mathfrak{g}]$-module generated by the "adjoint vector fields", i.e., vector fields ς of the form $\varsigma(y)(x) = [x, y]$, $x, y \in \mathfrak{g}$. A substantial generalisation of Dixmier's theorem was found by Levasseur and Stafford. They explicitly described the centraliser of $\Bbbk[\mathfrak{g}]^G$ in the algebra of differential operators on $\mathfrak{g}$. On the level of vector fields, their result reduces to Dixmier's theorem. The purpose of this paper is to explore similar problems in the general context of affine algebraic groups and their rational representations.

INTRODUCTION

We work over an algebraically closed field $\Bbbk$ of characteristic zero. Throughout, G is a connected affine algebraic group with Lie algebra $\mathfrak{g}$. Suppose for a while that G is semisimple. In 1979, Jacques Dixmier proved a nice theorem on vector fields on $\mathfrak{g}$. Specifically, he showed that any vector field annihilating all G-invariant polynomials on $\mathfrak{g}$ lies in the $\Bbbk[\mathfrak{g}]$-module generated by the "adjoint vector fields" [6, Theorem 2.1]. A substantial generalisation of Dixmier's theorem was found by Levasseur and Stafford [13]. They explicitly described the centraliser of $\Bbbk[\mathfrak{g}]^G$ in the algebra of differential operators on $\mathfrak{g}$. On the level of vector fields, their result reduces to Dixmier's theorem. The purpose of this paper is to explore similar problems in the general context of affine algebraic groups and their rational representations.

We show that Dixmier's argument applies to the coadjoint representations of the so-called '3-wonderful' Lie algebras. Furthermore, the coadjoint representation can be replaced with an arbitrary (finite-dimensional) representation, and this leads to three types of interesting problems. Let now G be an arbitrary connected group. We say that $\mathfrak{g}$ is *3-wonderful* if: (i) $\operatorname{codim}(\mathfrak{g}^* \setminus \mathfrak{g}^*_{reg}) \geqslant 3$, (ii) $\Bbbk[\mathfrak{g}^*]^G$ is a polynomial algebra of Krull dimension $\operatorname{ind} \mathfrak{g}$, and (iii) the sum of degrees of free homogeneous generators of $\Bbbk[\mathfrak{g}^*]^G$ equals $(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g})/2$. (Here $\mathfrak{g}^*_{reg}$ is the union of G-orbits of maximal dimension in $\mathfrak{g}^*$.)

This definition intends to axiomatise good properties of reductive Lie algebras. There is also a method for generating new 3-wonderful algebras: if $\mathfrak{g}$ is 3-wonderful and $\mathfrak{g} \simeq \mathfrak{g}^*$,

then the semi-direct product $\mathfrak{g} \ltimes \mathfrak{g}$ has the same properties. Below is a Dixmier-type result for the coadjoint representation of a 3-wonderful Lie algebra $\mathfrak{g}$.

Theorem 0.1. *Let $\mathfrak{X}$ be a polynomial vector field on $\mathfrak{g}^*$. Assume that $\mathfrak{X}$ annihilates all of $\Bbbk[\mathfrak{g}^*]^G$. Then there is a polynomial mapping $\mathfrak{Y} : \mathfrak{g}^* \to \mathfrak{g}$ such that $\mathfrak{X}(\xi) = \mathfrak{Y}(\xi){\cdot}\xi$ for any $\xi \in \mathfrak{g}^*$.*

The proof is essentially based on the fact that $\mathrm{codim}(\mathfrak{g}^* \setminus \mathfrak{g}^*_{reg}) \geqslant 3$ and certain vector bundle on $\mathfrak{g}^*_{reg}$ appears to be trivial. A posteriori, this is related to some good properties of a homomorphism of $\Bbbk[\mathfrak{g}^*]$-modules. Let $\mathsf{Mor}(V, N)$ denote the set of all polynomial morphisms $V \to N$, where V and N are $\Bbbk$-vector spaces. It is a free graded $\Bbbk[V]$-module of rank $\dim N$. Consider the homomorphism $\hat{\phi} : \mathsf{Mor}(\mathfrak{g}^*, \mathfrak{g}) \to \mathsf{Mor}(\mathfrak{g}^*, \mathfrak{g}^*)$, $\hat{\phi}(F)(\xi) := F(\xi){\cdot}\xi$ for $\xi \in \mathfrak{g}^*$. Then Theorem 0.1 merely says that $\mathrm{Im}\,\hat{\phi}$ equals the submodule of vector fields annihilating all of $\Bbbk[\mathfrak{g}^*]^G$. Furthermore, in the "3-wonderful case" the kernel of $\hat{\phi}$ appears to be a free $\Bbbk[\mathfrak{g}^*]$-module generated by G-equivariant morphisms.

For an arbitrary G-module V, where $\mathfrak{g}$ is not necessarily 3-wonderful, one can write up *three* similar homomorphisms and consider similar problems. The most obvious possibility is to replace $\mathfrak{g}^*$ with V. This yields the homomorphism of $\Bbbk[V]$-modules $\hat{\phi} : \mathsf{Mor}(V, \mathfrak{g}) \to \mathsf{Mor}(V, V)$. Clearly, any vector field $\mathfrak{X} \in \mathrm{Im}\,\hat{\phi}$ annihilates all of $\Bbbk[V]^G$. The problem on the opposite inclusion is related to the structure of $\mathrm{Ker}\,\hat{\phi}$, $\Bbbk[V]^G$, and V_{reg}, and we provide an appropriate analogue of Theorem 0.1. Two other possibilities are

$$\hat{\psi} : \mathsf{Mor}(V, V^*){\to}\mathsf{Mor}(V, \mathfrak{g}^*) \ \text{ and } \ \hat{\tau} : \mathsf{Mor}(\mathfrak{g}, V){\to}\mathsf{Mor}(\mathfrak{g}, V),$$

where we also describe the respective images under similar conditions, and give some illustrations.

Generalising the approach of [13], we regard the problem on $\mathrm{Im}\,\hat{\phi}$ as a special case of a problem on differential operators on V. Let $\mathcal{D}(V)$ denote the ring of differential operators on V with polynomial coefficients. As $\Bbbk[V]$ is identified with the differential operators of order zero, one can consider the centraliser of $\Bbbk[V]^G$ in $\mathcal{D}(V)$, $\mathsf{Cent}_{\mathcal{D}(V)}(\Bbbk[V]^G)$. Each $x \in \mathfrak{g}$ gives rise to a linear operator on V and therefore a vector field. In this way, one obtains the Lie algebra homomorphism

$$\varsigma = \varsigma_V : \mathfrak{g} \to \{\text{polynomial vector fields on } V\} \subset \mathcal{D}(V).$$

The elements of $\varsigma(\mathfrak{g})$ are called the *adjoint vector fields* (on V). By the definition of $\hat{\phi}$, $\mathrm{Im}\,\hat{\phi}$ is the $\Bbbk[V]$-module generated by $\varsigma(\mathfrak{g})$. Clearly, $\mathcal{C} := \mathsf{Cent}_{\mathcal{D}(V)}(\Bbbk[V]^G)$ contains $\Bbbk[V]$ and $\varsigma(\mathfrak{g})$, and one may ask whether the subalgebra generated by $\Bbbk[V]$ and $\varsigma(\mathfrak{g})$, denoted $\mathcal{A}$, is equal to $\mathcal{C}$. On the level of vector fields, the equality $\mathcal{A} = \mathcal{C}$ reduces to the assertion that any $\mathfrak{X}$ annihilating all of $\Bbbk[V]^G$ lies in $\mathrm{Im}\,\hat{\phi}$, i.e., in $\Bbbk[V]\varsigma(\mathfrak{g})$.

For the adjoint representation of a semisimple Lie algebra, the equality $\mathcal{A} = \mathcal{C}$ is proved in [13]. Adapting that method, we obtain a sufficient condition for $\mathcal{A} = \mathcal{C}$ in a more

general framework. We assume that $\Bbbk[V]^G$ is a polynomial algebra and $\operatorname{Ker}\hat{\phi}$ is a free $\Bbbk[V]$-module, and impose determinantal constraints on the embedding $\operatorname{Ker}\hat{\phi} \to \operatorname{Mor}(V, \mathfrak{g})$. However, the reductivity of G is not assumed. (See Theorem 4.2 for precise formulations). The equality $\mathcal{A} = \mathcal{C}$ and other results on $\mathcal{C}$ stem from assertions about certain G-stable subvariety of $V \times \mathfrak{g}^*$. Recall that for any G-module V, there is the moment map $\mu :$ $V \times V^* \to \mathfrak{g}^*$. Then $\varkappa : V \times V^* \to V \times \mathfrak{g}^*$ is defined by letting $\varkappa(v, \xi) = (v, \mu(v, \xi))$. Consider also the $\Bbbk[V]$-module $E = \operatorname{Im}\hat{\phi}$ and its symmetric algebra, $\mathsf{Sym}_{\Bbbk[V]}(E)$. Under appropriate constraints (alluded to above), we prove that

- $\mathsf{Sym}_{\Bbbk[V]}(E)$ is a factorial domain of Krull dimension $\dim V + \dim \mathfrak{g} - \operatorname{rk}(\operatorname{Ker}\hat{\phi})$;
- $\overline{\operatorname{Im}\varkappa} = \operatorname{Spec}(\mathsf{Sym}_{\Bbbk[V]}(E))$ and it is also a complete intersection in $V \times \mathfrak{g}^*$;
- the generators of the ideal of $\overline{\operatorname{Im}\varkappa}$ are determined by a basis of $\operatorname{Ker}\hat{\phi}$.

From this, we deduce that $\operatorname{gr}\mathcal{A} = \operatorname{gr}\mathcal{C} = \mathsf{Sym}_{\Bbbk[V]}(E)$, where $\operatorname{gr}(.)$ is the associated graded ring with respect to the filtration by the order of differential operators. Then the equality $\mathcal{A} = \mathcal{C}$ follows. We also give a sufficient condition for $\mathcal{C}$ to be a free $\Bbbk[V]^G$-module (see Theorem 4.9).

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a $\mathbb{Z}_2$-graded semisimple Lie algebra. This grading (or the symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$) is said to be $\mathcal{N}$-*regular*, if $\mathfrak{g}_1$ contains a regular nilpotent element of $\mathfrak{g}$. Our main application concerns the isotropy representation $(G_0 : \mathfrak{g}_1)$. We show that the hypotheses of Theorems 4.2 and 4.9 are satisfied, modulo one exception, for $(G_0 : \mathfrak{g}_1)$ if $(\mathfrak{g}, \mathfrak{g}_0)$ is $\mathcal{N}$-regular. Hence $\mathcal{A} = \mathcal{C}$ and $\mathcal{C}$ is a free $\Bbbk[\mathfrak{g}_1]^{G_0}$-module in these cases. Verification of all necessary conditions requires a detailed information on the structure of the null-cone in the G_0-module $\mathfrak{g}_1$. We also provide other examples of the "$\mathcal{A} = \mathcal{C}$ phenomenon"; in particular, those for the coadjoint representation of non-reductive Lie algebras.

The plan of the article is as follows. Section 1 contains preliminaries on group actions and differential operators. In Section 2, we prove our analogue of Dixmier's result for the coadjoint representation of a 3-wonderful Lie algebra. Then we discuss, in Section 3, three generalisations to the case in which $\mathfrak{g}^*$ is replaced with an arbitrary G-module. Section 4 contains our results on the image of $\varkappa$, $\mathsf{Sym}_{\Bbbk[V]}(E)$, and the equality $\mathcal{A} = \mathcal{C}$. In Section 5, we consider applications to $\mathbb{Z}_2$-gradings of semisimple Lie algebras and provide some other examples. In Section 6, we discuss possible connections between our results for $\mathbb{Z}_2$-gradings and another generalisation of Dixmier's result obtained in [12, 14].

Some notation. If an algebraic group G acts on an irreducible affine variety X, then $\Bbbk[X]^G$ is the algebra of G-invariant regular functions on X and $\Bbbk(X)^G$ is the field of G-invariant rational functions. If $\Bbbk[X]^G$ is finitely generated, then $X /\!\!/ G := \operatorname{Spec}\Bbbk[X]^G$, and the *quotient morphism* $\pi_X : X \to X /\!\!/ G$ is the mapping associated with the embedding $\Bbbk[X]^G \hookrightarrow \Bbbk[X]$. We use dot '$\cdot$' to denote the action of (elements of) G and $\mathfrak{g}$ on X. For instance, $G{\cdot}x$ is the orbit of $x \in X$. The stabiliser of x in $\mathfrak{g}$ is denoted by $\mathfrak{g}_x$.

All topological terms refer to the Zariski topology. The pairing of dual vector spaces is denoted by $\langle\,,\,\rangle$. If M is a subset of a vector space, then $\mathsf{span}(M)$ denotes the linear span of M.

1. Preliminaries

1.1.   Let G be an affine algebraic group acting regularly on an irreducible algebraic variety X. We say that $\mathfrak{h} \subset \mathfrak{g}$ is *a generic stabiliser* for the the action $(G : X)$ if there exists a dense open subset $\Omega \subset X$ such that all stabilisers $\mathfrak{g}_x$, $x \in \Omega$, are G-conjugate to $\mathfrak{h}$. The points of such an Ω are said to be *generic*. Generic stabilisers always exist if G is reductive and X is smooth [21].

Let X_{reg} denote the set of all *regular* elements of X. That is,

$$X_{reg} := \{x \in X \mid \dim G{\cdot}x \geqslant \dim G{\cdot}x' \text{ for all } x' \in X\} =$$
$$\{x \in X \mid \dim \mathfrak{g}_x \leqslant \dim \mathfrak{g}_{x'} \text{ for all } x' \in X\}\,.$$

As is well-known, X_{reg} is a dense open subset of X. If we want to explicitly specify the group acting on X, we refer to *G-regular* elements.

Definition 1. A G-variety X is said to have the *codim–n property* if $\mathrm{codim}_X(X \setminus X_{reg}) \geqslant n$.

We will mostly use this notion if $X = V$ is a G-module.

Example. Let $\mathfrak{g}$ be reductive and $\mathcal{N} \subset \mathfrak{g}$ the nilpotent cone. Then $\mathfrak{g}$ (resp. $\mathcal{N}$) has the codim–3 (resp. codim–2) property with respect to the adjoint representation [9].

Recall that the *index* of $\mathfrak{g}$, denoted $\mathrm{ind}\,\mathfrak{g}$, is the minimal dimension of stabilisers for the elements of the $\mathfrak{g}$-module $\mathfrak{g}^*$. That is, $\mathrm{ind}\,\mathfrak{g} = \min\limits_{\xi \in \mathfrak{g}^*} \dim \mathfrak{g}_\xi = \dim \mathfrak{g}_\eta$ for any $\eta \in \mathfrak{g}^*_{reg}$.

1.2.   For finite-dimensional $\Bbbk$-vector spaces V and N, let $\mathsf{Mor}(V, N)$ denote the set of *polynomial* morphisms $V \to N$. Clearly, $\mathsf{Mor}(V, N) \simeq \Bbbk[V] \otimes N$ and it is a free graded $\Bbbk[V]$-module of rank $\dim N$.

If V and N are G-modules, then G acts on $\mathsf{Mor}(V, N)$ by the rule $(g{*}F)(v) = g{\cdot}(F(g^{-1}{\cdot}v))$. Then $(\mathsf{Mor}(V, N))^G =: \mathsf{Mor}_G(V, N)$ is the set of all polynomial G-equivariant morphisms $V \to N$. It is a $\Bbbk[V]^G$-module, which is called *the module of covariants of type N*. If G is reductive, then the algebra $\Bbbk[V]^G$ is finitely generated and each $\mathsf{Mor}_G(V, N)$ is a finitely generated $\Bbbk[V]^G$-module.

[All these constructions makes sense if V is replaced with any affine G-variety X.]

1.3. Let $\mathcal{D}(V)$ denote the algebra of differential operators on V, with polynomial coefficients. Recall that $\mathcal{D}(V)$ contains the symmetric algebra of V, $\mathcal{S}(V)$, as the subalgebra of constant coefficient differential operators and $\Bbbk[V]$ as the subalgebra of differential operators of order zero. We always filter $\mathcal{D}(V)$ by the order of differential operators, hence $\mathrm{gr}_n \mathcal{D}(V) \simeq \Bbbk[V] \otimes \mathcal{S}^n(V)$ and $\mathrm{gr}\,\mathcal{D}(V)$ is isomorphic to $\Bbbk[V] \otimes \mathcal{S}(V) = \Bbbk[V \times V^*]$ as algebras.

Let $\mathrm{Der}(\Bbbk[V])$ denotes the $\Bbbk[V]$-module of all $\Bbbk$-derivations of $\Bbbk[V]$ or, equivalently, the module of polynomial vector fields on V. Then $\mathrm{Der}(\Bbbk[V]) \simeq \mathrm{Mor}(V, V)$. A vector field $\mathfrak{X}$ can be regarded either as polynomial endomorphism of V or as linear endomorphism of $\Bbbk[V]$. The respective notation is $\mathfrak{X}(v)$, $v \in V$ and $\mathfrak{X}\{f\}$, $f \in \Bbbk[V]$.

2. ADJOINT VECTOR FIELDS AND 3-WONDERFUL LIE ALGEBRAS

In this section, G is a connected algebraic group.

Let V be a (finite-dimensional, rational) G-module. The differential of the G-action on $\Bbbk[V]$ yields a map $\varsigma = \varsigma_V : \mathfrak{g} \to \mathrm{Der}(\Bbbk[V]) \subset \mathcal{D}(V)$. Upon the identification $\mathrm{Der}(\Bbbk[V])$ with $\mathrm{Mor}(V, V)$, we see that $\varsigma(e)$ is just the linear operator on V corresponding to $e \in \mathfrak{g}$. The vector fields on V of the form $\varsigma(e)$ are said to be the *adjoint vector fields*. For $\mathfrak{g}$ semisimple and $V = \mathfrak{g}$, Dixmier describes a relationship between the adjoint vector fields and vector fields annihilating all of $\Bbbk[\mathfrak{g}]^G$ [6, Theorem 2.1]. Below, we prove that this result naturally extends to the coadjoint representations of certain non-reductive Lie algebras.

In [9], Kostant established a number of fundamental properties of complex reductive Lie algebras. Motivated by these results, we give the following

Definition 2. An algebraic Lie algebra $\mathfrak{g}$ is said to be *n-wonderful*, if the following conditions are satisfied:

   (i) the coadjoint representation of $\mathfrak{g}$ has the codim–n property.
  (ii) $\Bbbk[\mathfrak{g}^*]^G$ is a polynomial algebra of Krull dimension $l = \mathrm{ind}\,\mathfrak{g}$;
 (iii) If $f_1, \ldots, f_l$ are homogeneous algebraically independent generators of $\Bbbk[\mathfrak{g}^*]^G$, then
$$\sum_{i=1}^{l} \deg f_i = (\dim \mathfrak{g} + \mathrm{ind}\,\mathfrak{g})/2;$$

Remark 2.1. We are only interested in n-wonderful algebras for $n = 2, 3$. Let us point out some connections between hypotheses of this definition, and their consequences.

1. For any Lie algebra, $\mathrm{trdeg}\,\Bbbk(\mathfrak{g}^*)^G$ equals $\mathrm{ind}\,\mathfrak{g}$ and hence $\Bbbk[\mathfrak{g}^*]^G$ contains at most $\mathrm{ind}\,\mathfrak{g}$ algebraically independent elements. Thus, condition (ii) also means that $(\mathfrak{g}, \mathrm{ad}^*)$ has sufficiently many polynomial invariants.

2. If $(\mathfrak{g}, \mathrm{ad}^*)$ has the codim–2 property and $f_1, \ldots, f_l \in \Bbbk[\mathfrak{g}^*]^G$ are algebraically independent, then $\sum_{i=1}^{l} \deg f_i \geqslant (\dim \mathfrak{g} + \mathrm{ind}\,\mathfrak{g})/2$. Furthermore, if the equality holds, then $f_1, \ldots, f_l$ freely generate $\Bbbk[\mathfrak{g}^*]^G$ and

(2.1) $$\mathfrak{g}_{reg}^* = \{\xi \in \mathfrak{g}^* \mid (df_1)_\xi, \ldots, (df_l)_\xi \text{ are linearly independent}\},$$

see [18, Theorem 1.2]. It follows that Eq. (2.1) holds for any 2-wonderful algebra. For $\mathfrak{g}$ reductive, equality (2.1) is a celebrated result of Kostant [9, Theorem 0.1].

3. The main result of [20] asserts that if $\mathfrak{g}$ is 3-wonderful, then the Poisson commutative subalgebra of $\Bbbk[\mathfrak{g}^*]$ obtained from $\Bbbk[\mathfrak{g}^*]^G$ via the argument shift method is *maximal* for any $\xi \in \mathfrak{g}^*_{reg}$.

4. Any reductive Lie algebra is 3-wonderful. Several non-trivial examples of 3-wonderful algebras are discussed in [20, Section 4].

Theorem 2.2. *Let $\mathfrak{g}$ be a 3-wonderful Lie algebra. Given a polynomial vector field $\mathfrak{X}$ on $\mathfrak{g}^*$, the following conditions are equivalent:*

(i) *$\mathfrak{X}$ annihilates all G-invariant polynomials on $\mathfrak{g}^*$;*
(ii) *$\mathfrak{X}(\xi) \in \mathfrak{g}{\cdot}\xi$ for any $\xi \in \mathfrak{g}^*_{reg}$;*
(iii) *$\mathfrak{X}(\xi) \in \mathfrak{g}{\cdot}\xi$ for any $\xi \in \mathfrak{g}^*$;*
(iv) *There is a polynomial mapping $\mathfrak{Y} \in \mathsf{Mor}(\mathfrak{g}^*, \mathfrak{g})$ such that $\mathfrak{X}(\xi) = \mathfrak{Y}(\xi){\cdot}\xi$ for any $\xi \in \mathfrak{g}^*$.*

Proof. Recall that for $f \in \Bbbk[\mathfrak{g}^*]$ the polynomial $\mathfrak{X}\{f\} \in \Bbbk[\mathfrak{g}^*]$ is defined by

$$(2.2) \qquad\qquad \mathfrak{X}\{f\}(\xi) = \langle \mathfrak{X}(\xi), (df)_\xi \rangle.$$

It is therefore clear that (iv)$\Rightarrow$(iii)$\Rightarrow$(ii)$\Rightarrow$(i). It remains to prove the implication (i)$\Rightarrow$(iv). To this end, we need some preparations. Up to some obvious alterations, the rest of the proof is a repetition of the proof of Theorem 2.1 in [6].

Set $\Omega = \mathfrak{g}^*_{reg}$. If $\xi \in \Omega$, then $(df_1)_\xi, \ldots, (df_l)_\xi$ form a basis for $\mathfrak{g}_\xi$, in view of Definition 2 and Eq. (2.1).

Let E be the cotangent bundle of Ω, which is identified with $E \simeq \Omega \times \mathfrak{g}$. Let E' be the sub-bundle of E whose fibre of ξ is $\mathfrak{g}_\xi$. The previous paragraph shows that the df_i's yield a trivialisation of E'. Let E'' be the sub-bundle of the tangent bundle of Ω whose fibre of ξ is $\mathfrak{g}{\cdot}\xi$. Since the kernel of the surjective mapping $(x \in \mathfrak{g}) \mapsto (x{\cdot}\xi \in \mathfrak{g}{\cdot}\xi)$ is $\mathfrak{g}_\xi$, one obtains the exact sequence of vector bundles

$$0 \to E' \to E \to E'' \to 0$$

and the exact sequence

$$H^0(\Omega, E) \to H^0(\Omega, E'') \to H^1(\Omega, E') \,.$$

Let $\mathcal{O}$ denote the structure sheaf of $\mathfrak{g}^*$. By [5, cor. 2.9, p.16], there exists an exact sequence of cohomology groups

$$H^1(\mathfrak{g}^*, \mathcal{O}) \to H^1(\Omega, \mathcal{O}|_\Omega) \to H^2_{\mathfrak{g}^* \backslash \Omega}(\mathfrak{g}^*, \mathcal{O}) \,.$$

Here $H^1(\mathfrak{g}^*, \mathcal{O}) = 0$ because $\mathfrak{g}^*$ is affine, and it follows from the codim–3 property that $H^2_{\mathfrak{g}^* \backslash \Omega}(\mathfrak{g}^*, \mathcal{O}) = 0$ [5, cor. 1.4, p.80]. Hence $H^1(\Omega, \mathcal{O}|_\Omega) = 0$. This fact and the trivilality of E' imply that $H^1(\Omega, E') = 0$. Thus, the homomorphism $\gamma : H^0(\Omega, E) \to H^0(\Omega, E'')$ is onto.

Suppose that $\mathfrak{X}$ satisfies assumption (i). Then Eq. (2.2) and the linear independence of the differentials $(df_i)_\xi$, $\xi \in \Omega$, show that $\mathfrak{X}$ also satisfies (ii). Therefore $\mathfrak{X}|_\Omega$ is a section of E''. The surjectivity of γ means that there exists a polynomial mapping $\mathfrak{Y}_0 : \Omega \to \mathfrak{g}$ such that $\mathfrak{X}(\xi) = \mathfrak{Y}_0(\xi) \cdot \xi$ for any $\xi \in \Omega$. Since $\mathrm{codim}(\mathfrak{g}^* \setminus \Omega) \geqslant 2$, $\mathfrak{Y}_0$ extends to a polynomial mapping $\mathfrak{Y} : \Omega \to \mathfrak{g}$, and the equality $\mathfrak{X}(\xi) = \mathfrak{Y}(\xi) \cdot \xi$ holds for all $\xi \in \mathfrak{g}^*$. $\qquad \square$

Remark 2.3. This theorem is a statement about the coadjoint representation of G. There are two key points in the proof. First, E' appears to be a trivial bundle and, second, $\mathrm{codim}(\mathfrak{g}^* \setminus \Omega) \geqslant 3$. Using this observation, we show in Section 3 that Theorem 2.2 admits various generalisations to other representations of G.

Remark 2.4. We know that $\mathrm{Der}(\Bbbk[\mathfrak{g}^*])$ is a $\Bbbk[\mathfrak{g}^*]$-module and $\varsigma(\mathfrak{g}) \subset \mathrm{Der}(\Bbbk[\mathfrak{g}^*])$, where $\varsigma = \varsigma_{\mathfrak{g}^*}$. Therefore, implication (i)$\Rightarrow$(iv) in Theorem 2.2 can be stated as follows:

If $\mathfrak{X} \in \mathrm{Der}(\Bbbk[\mathfrak{g}^*])$ annihilates all of $\Bbbk[\mathfrak{g}^*]^G$, then $\mathfrak{X} \in \Bbbk[\mathfrak{g}^*] \cdot \varsigma(\mathfrak{g})$.

Remark 2.5. For an arbitrary Lie algebra $\mathfrak{g}$, define the homomorphism of $\Bbbk[\mathfrak{g}^*]$-modules $\hat{\phi} : \mathrm{Mor}(\mathfrak{g}^*, \mathfrak{g}) \to \mathrm{Mor}(\mathfrak{g}^*, \mathfrak{g}^*)$ by $\hat{\phi}(F)(\xi) = F(\xi) \cdot \xi$, $\xi \in \mathfrak{g}^*$. Then

$$\mathrm{Im}\, \hat{\phi} \subset \{\mathcal{F} : \mathfrak{g}^* \to \mathfrak{g}^* \mid \mathcal{F}(\xi) \in \mathfrak{g} \cdot \xi \quad \forall \xi \in \mathfrak{g}^*\} =: \mathcal{T}.$$

Since the elements of $\mathrm{Mor}(\mathfrak{g}^*, \mathfrak{g}^*)$ is are just the vector field on $\mathfrak{g}^*$, the equivalence of conditions (iii) and (iv) in Theorem 2.2 reduces to the assertion that if $\mathfrak{g}$ is 3-wonderful, then $\mathrm{Im}\, \hat{\phi} = \mathcal{T}$.

Notice that $\mathrm{Ker}\, \hat{\phi} = \{F : \mathfrak{g}^* \to \mathfrak{g} \mid F(\xi) \in \mathfrak{g}_\xi \quad \forall \xi \in \mathfrak{g}^*\}$. If $\mathrm{Ker}\, \hat{\phi}$ is a *free* $\Bbbk[\mathfrak{g}^*]$-module (of rank $l = \mathrm{ind}\, \mathfrak{g}$) and $F_1, \ldots, F_l$ is a basis, then E' is a trivial vector bundle over $\Omega' = \{\xi \in \mathfrak{g}^*_{reg} \mid F_1(\xi), \ldots, F_l(\xi) \text{ are linearly independent}\}$. For any $f \in \Bbbk[\mathfrak{g}^*]^G$, we have $df \in \mathrm{Ker}\, \hat{\phi} \cap \mathrm{Mor}_G(\mathfrak{g}^*, \mathfrak{g})$. If $\mathfrak{g}$ is 2-wonderful, then [15, Theorem 1.9] applies to the coadjoint representation of $\mathfrak{g}$, and one concludes that $\mathrm{Ker}\, \hat{\phi}$ is freely generated by the differentials $df_1, \ldots, df_l$. Then, using Eq. (2.1), we obtain $\Omega' = \mathfrak{g}^*_{reg}$. This argument shows that in some cases (actually, most interesting ones), the triviality of E' is closely related to the fact that $\mathrm{Ker}\, \hat{\phi}$ is a free $\Bbbk[\mathfrak{g}^*]$-module generated by G-equivariant morphisms.

Example 2.6. There is a procedure that generates new n-wonderful algebras from old ones (for $n \geqslant 2$). Let $\mathfrak{q}$ be a quadratic n-wonderful Lie algebra ("quadratic" means that $\mathfrak{q}^* \simeq \mathfrak{q}$ as $\mathfrak{q}$-module) . Form the semi-direct product $\mathfrak{g} = \mathfrak{q} \ltimes \mathfrak{q}$ (the second copy of $\mathfrak{q}$ is an Abelian ideal of $\mathfrak{g}$). Then $\mathfrak{g}$ is again quadratic and n-wonderful. That $\mathfrak{g}$ to be quadratic is elementary. Therefore we can deal with the adjoint representation of $\mathfrak{g}$. It then suffices to apply Theorem 7.1 in [17] to the case $V = \mathfrak{q}$. Roughly speaking, that theorem says that

the passage $\mathfrak{q} \mapsto \mathfrak{q} \ltimes \mathfrak{q}$ doubles all data occurring in Definition 2. That is, $\dim \mathfrak{g} = 2 \dim \mathfrak{q}$, $\operatorname{ind} \mathfrak{g} = 2 \operatorname{ind} \mathfrak{q}$; each basis invariant $f_i \in \Bbbk[\mathfrak{q}]^Q$ gives rise to two basis invariants in $\Bbbk[\mathfrak{g}]^G$, and the degree for all three is the same. Finally, it is easily seen that $\mathfrak{q}_{reg} \ltimes \mathfrak{q} \subset \mathfrak{g}_{reg}$. Hence the codim–$n$ property is also preserved.

In particular, one can start with any semisimple $\mathfrak{s}$ and take $\mathfrak{s} \ltimes \mathfrak{s}$. This yields interesting examples of 3-wonderful algebras. Notice that then this procedure can be iterated ad infinum.

3. Modules over polynomial rings associated with representations

Unless otherwise stated, G is an arbitrary connected algebraic group. Let V be a G-module. Associated with V, $\mathfrak{g}$, and $\mathfrak{g}^*$, there are at least three natural exact sequences of modules over polynomial rings:

(A) $0 \to \operatorname{Ker} \hat{\phi} \to \operatorname{Mor}(V, \mathfrak{g}) \xrightarrow{\hat{\phi}} \operatorname{Mor}(V, V)$,

(B) $0 \to \operatorname{Ker} \hat{\psi} \to \operatorname{Mor}(V, V^*) \xrightarrow{\hat{\psi}} \operatorname{Mor}(V, \mathfrak{g}^*)$,

(C) $0 \to \operatorname{Ker} \hat{\tau} \to \operatorname{Mor}(\mathfrak{g}, V) \xrightarrow{\hat{\tau}} \operatorname{Mor}(\mathfrak{g}, V)$.

The first two sequences consist of $\Bbbk[V]$-modules, and the last one consists of $\Bbbk[\mathfrak{g}]$-modules. Some of the properties of (A) and (B) have been studied in [15], whereas (B) and (C) have also been considered in [17, Sect. 8]. Recall the definitions of $\hat{\phi}, \hat{\psi}, \hat{\tau}$:

(A) $\hat{\phi}(F)(v) := F(v){\cdot}v$, where $v \in V$;

(B) $\langle \hat{\psi}(F)(v), x \rangle := \langle x{\cdot}v, F(v) \rangle$, where $v \in V$, $x \in \mathfrak{g}$, and $\langle \, , \, \rangle$ stands for the pairing of elements of dual vector spaces. One can also exploit the *moment mapping* $\mu : V \times V^* \to \mathfrak{g}^*$, which is defined by $\langle \mu(v, \eta), x \rangle = \langle x{\cdot}v, \eta \rangle$, where $\eta \in V^*$. Then $\hat{\psi}(F)(v) := \mu(v, F(v))$.

(C) $\hat{\tau}(F)(x) := x{\cdot}F(x)$, where $x \in \mathfrak{g}$.

Remark. $\operatorname{Ker} \hat{\phi}$ is a G-stable submodule of $\operatorname{Mor}(V, \mathfrak{g})$; and likewise for $\operatorname{Ker} \hat{\psi}$ and $\operatorname{Ker} \hat{\tau}$.

Note that, for $V = \mathfrak{g}^*$, the sequences (A) and (B) coincide, and we obtain the situation of Remark 2.5. Also, the sequences (A) and (C) coincide if $V = \mathfrak{g}$. Below we formulate Dixmier-type statements, which characterise the images of $\hat{\phi}, \hat{\psi}$, and $\hat{\tau}$ under similar (rather restrictive) assumptions.

Case (A). Here $\operatorname{Ker} \hat{\phi} = \{F : V \to \mathfrak{g} \mid F(v) \in \mathfrak{g}_v \ \forall v \in V\}$ and $\operatorname{Im} \hat{\phi} \subset \{\mathcal{F} : V \to V \mid \mathcal{F}(v) \in \mathfrak{g}{\cdot}v \ \forall v \in V\}$. Set $\Omega_\phi = V_{reg}$. Consider three vector bundles on Ω_ϕ:

$$E'_\phi = \{(v, x) \mid x{\cdot}v = 0\} = \{(v, x)) \mid x \in \mathfrak{g}_v\} \subset \Omega_\phi \times \mathfrak{g},$$

$$E_\phi = \Omega_\phi \times \mathfrak{g}, \quad E''_\phi = \{(v, x{\cdot}v) \mid v \in \Omega_\phi, \ x \in \mathfrak{g}\} \subset \Omega_\phi \times V$$

and the corresponding exact sequence $0 \to E'_\phi \to E_\phi \to E''_\phi \to 0$. Arguing as in the proof of Theorem 2.2, one obtains

Proposition 3.1. *Suppose E'_ϕ is a trivial vector bundle and $\mathrm{codim}(V \setminus \Omega_\phi) \geqslant 3$. Then*

$$\mathrm{Im}\,\hat{\phi} = \{\mathcal{F} : V \to V \mid \mathcal{F}(v) \in \mathfrak{g}\cdot v \ \ \forall v \in V\}.$$

In other words, if $\mathcal{F}(v) \in \mathfrak{g}\cdot v$ for all $v \in \Omega_\phi$, then there is $F : V \to \mathfrak{g}$ such that $\mathcal{F}(v) = F(v)\cdot v$ for all $v \in V$.

This is not a complete analogue of Theorem 2.2, since we obtain only equivalence of the following three conditions on the vector field $\mathcal{F} : V \to V$:

- (ii) $\mathcal{F}(v) \in \mathfrak{g}\cdot v$ for any $v \in V_{reg}$;
- (iii) $\mathcal{F}(v) \in \mathfrak{g}\cdot v$ for any $v \in V$;
- (iv) There is an $F \in \mathrm{Mor}(V, \mathfrak{g})$ such that $\mathcal{F}(v) = F(v)\cdot\xi$ for any $\xi \in V$.

In order to add condition

- (i) $\mathcal{F}$ annihilates all of $\Bbbk[V]^G$

to this list, one has to impose some constraints on $\Bbbk[V]^G$. For instance, it suffices to require that the quotient field of $\Bbbk[V]^G$ equals to $\Bbbk(V)^G$ and that $\dim(\mathrm{span}\{df_v \mid f \in \Bbbk[V]^G\}) = \mathrm{trdeg}\,\Bbbk(V)^G$ for any $v \in V_{reg}$. (Cf. the proof of Theorem 2.2). Actually, these two conditions are not too restrictive. These are always satisfied if G is semisimple and $\Bbbk[V]^G$ is a polynomial (free) algebra (see [8]).

The problem of triviality for E'_ϕ is connected with the question of whether $\mathrm{Ker}\,\hat{\phi}$ is a free $\Bbbk[V]$-module. This seems to be related to the property that a generic stabiliser for $(\mathfrak{g} : V)$ is abelian. In the following sections, we study case (A) more carefully, prove a more general result, and provide some examples.

Case (B). Here $\mathrm{Ker}\,\hat{\psi} = \{F : V \to V^* \mid \mu(v, F(v)) = 0 \ \ \forall v \in V\}$ and $\mathrm{Im}\,\hat{\psi} \subset \{\mathcal{F} : V \to \mathfrak{g}^* \mid \mathcal{F}(v) \in \mu(v, V^*) \ \forall v \in V\}$. Again, we take $\Omega_\psi = V_{reg}$. Consider three vector bundles on Ω_ψ:

$$E'_\psi = \{(v, \xi) \mid \mu(v, \xi) = 0\} = \{(v, \xi) \mid \xi \in (\mathfrak{g}\cdot v)^\perp\} \subset \Omega_\psi \times V^*,$$
$$E_\psi = \Omega_\psi \times V^*, \quad E''_\psi = \{(v, \mu(v, \xi)) \mid v \in \Omega_\psi, \ \xi \in V^*\} \subset \Omega_\psi \times \mathfrak{g}^*$$

and the corresponding exact sequence $0 \to E'_\psi \to E_\psi \to E''_\psi \to 0$. Arguing as in the proof of Theorem 2.2, one obtains

Proposition 3.2. *Suppose E'_ψ is a trivial vector bundle and $\mathrm{codim}(V \setminus \Omega_\psi) \geqslant 3$. Then*

$$\mathrm{Im}\,\hat{\psi} = \{\mathcal{F} : V \to \mathfrak{g}^* \mid \mathcal{F}(v) \in \mu(v, V^*) \ \forall v \in V\}.$$

In other words, if $\mathcal{F}(v) \in \mu(v, V^*)$ for all $v \in \Omega_\psi$, then there is $F : V \to V^*$ such that $\mathcal{F}(v) = \mu(v, F(v))$ for all $v \in V$.

The hypotheses of Proposition 3.2 are satisfied in the following situation.

Theorem 3.3. *Suppose $\mathfrak{g}$ is semisimple, $\Bbbk[V]^G$ is polynomial, and $\mathrm{codim}(V \setminus V_{reg}) \geqslant 3$. Then $\mathrm{Ker}\,\hat{\psi}$ is a free $\Bbbk[V]$-module and Proposition 3.2 applies.*

Proof. Let $f_1, \ldots, f_l \in \Bbbk[V]^G$ be the basis invariants. By [15, 1.9 & 1.10], $\mathrm{Ker}\,\hat{\psi}$ is a free $\Bbbk[V]$-module generated by $df_1, \ldots, df_l$. By [8, Korollar 1], $V_{reg} \subset \{v \in V \mid (df_1)_v, \ldots, (df_l)_v \text{ are linearly independent}\}$. It follows that E'_ψ is a trivial bundle on V_{reg}.   □

It follows from [20, Remark 4.5] that, under the assumptions of Theorem 3.3, a generic stabiliser for $(G : V)$ has to be non-trivial, i.e., $\max \dim G{\cdot}v < \dim V$.

Case (C). For $x \in \mathfrak{g}$, we set $V^x = \{v \in V \mid x{\cdot}v = v\}$. Here $\mathrm{Ker}\,\hat{\tau} = \{F : \mathfrak{g} \to V \mid F(x) \in V^x \;\; \forall x \in \mathfrak{g}\}$ and $\mathrm{Im}\,\hat{\tau} \subset \{\mathcal{F} : \mathfrak{g} \to V \mid \mathcal{F}(x) \in x{\cdot}V \;\; \forall x \in \mathfrak{g}\}$. Set $\Omega_\tau = \{x \in \mathfrak{g} \mid \dim V^x \text{ is minimal}\}$. Consider three vector bundles on Ω_τ:

$$E'_\tau = \{(x,v) \mid x{\cdot}v = 0\} = \{(x,v) \mid v \in V^x\} \subset \Omega_\tau \times V, \quad E_\tau = \Omega_\tau \times V,$$

$$E''_\tau = \{(x, x{\cdot}v) \mid x \in \Omega_\tau, \; v \in V\} \subset \Omega_\tau \times V$$

and the corresponding exact sequence $0 \to E'_\tau \to E_\tau \to E''_\tau \to 0$. Arguing as in the proof of Theorem 2.2, one obtains

Proposition 3.4. *Suppose E'_τ is a trivial vector bundle and $\mathrm{codim}(\mathfrak{g} \setminus \Omega_\tau) \geqslant 3$. Then*

$$\mathrm{Im}\,\hat{\tau} = \{\mathcal{F} : \mathfrak{g} \to V \mid \mathcal{F}(v) \in x{\cdot}V \;\; \forall x \in \mathfrak{g}\}.$$

In other words, if $\mathcal{F}(x) \in x{\cdot}V$ for all $x \in \Omega_\tau$, then there is $F : \mathfrak{g} \to V$ such that $\mathcal{F}(x) = x{\cdot}F(x)$ for all $x \in q$.

It is remarkable that if G is reductive, then $\mathrm{Ker}\,\hat{\tau}$ is always a free $\Bbbk[\mathfrak{g}]$-module [17, Theorem 8.6]. There is also a special case in which all the assumptions of Proposition 3.4 are satisfied.

Theorem 3.5. *Let $\mathfrak{g}$ be reductive, $\mathfrak{t} \subset \mathfrak{g}$ a Cartan subalgebra, and $e \in \mathfrak{g}$ a regular nilpotent element. Suppose that*

$$(\diamond) \hspace{4cm} \dim V^e = \dim V^\mathfrak{t}.$$

Then $\Omega_\tau \supset \mathfrak{g}_{reg}$ and if $\mathcal{F}(x) \in x{\cdot}V$ for all $x \in \mathfrak{g}_{reg}$, then there is $F : \mathfrak{g} \to V$ such that $\mathcal{F}(x) = x{\cdot}F(x)$ for all $x \in \mathfrak{g}$.

Proof. It easily follows from assumption $(\diamond)$ that $\dim V^z = \dim V^\mathfrak{t}$ for any regular semisimple $z \in \mathfrak{g}$. Therefore the minimal value of $\dim V^x$ is the dimension of the zero-weight space of V, which is positive. That is, the open subset Ω_τ contains the regular semisimple and nilpotent elements. It follows that $\Omega_\tau \supset \mathfrak{g}_{reg}$. By [9], $\mathrm{codim}(\mathfrak{g} \setminus \mathfrak{g}_{reg}) = 3$. For triviality E'_τ, it is enough to prove that $\mathrm{Ker}\,\hat{\tau}$ is a free $\Bbbk[\mathfrak{g}]$-module, and the latter has been done in [17, Theorem 8.6].   □

Remark 3.6. The above equality $\dim V^z = \dim V^{\mathfrak{t}}$ (with z semisimple) means that each nonzero weight of V (with respect to $\mathfrak{t}$) is a multiple of a root. Using this observation, one easily obtains the complete list of irreducible representations of simple Lie algebras satisfying assumption ($\diamondsuit$). Here it is:

- the adjoint representation of $\mathfrak{g}$;
- $(\mathbf{B}_n, \varphi_1)$, $(\mathbf{B}_n, 2\varphi_1)$, $(\mathbf{C}_n, \varphi_2)$, $(\mathbf{F}_4, \varphi_1)$, $(\mathbf{G}_2, \varphi_1)$, $(\mathbf{G}_2, 2\varphi_1)$;
- $(\mathbf{A}_1, 2m\varphi_1)$ for any $m \in \mathbb{N}$.

Actually, each of the cases (A), or (B), or (C) deserves a special thorough treatment. In the following sections, we concentrate on case (A), partly in view of its connections with differential operators. Another reason is that similar properties of sequences (B) and (C) for representations of reductive groups have been studied in [17, Section 8].

4. Differential operators and invariant polynomials

In Section 3, three possibilities to generalise Dixmier's results have been discussed. These are related to three sequences of modules over polynomial rings. It seems that case (A) is the most interesting one, because the problem can further be transferred to the setting of differential operators on V.

The discussion of case (A) in Section 3 shows that if a G-module V satisfies certain explicit conditions, then a vector field $\mathcal{F} : V \to V$ annihilates all of $\Bbbk[V]^G$ if and only if there is $F \in \mathsf{Mor}(V, \mathfrak{g})$ such that $\mathcal{F}(v) = F(v) \cdot v$ for all $v \in V$. In other words,

$$(\diamondsuit) \qquad \mathcal{F}\{f\} = 0 \text{ for any } f \in \Bbbk[V]^G \text{ if and only if } \mathcal{F} \in \Bbbk[V]\varsigma(\mathfrak{g}).$$

(cf. Remark 2.4). Let us restate ($\diamondsuit$) using the algebra of differential operators $\mathcal{D}(V)$.

Let $\mathcal{C} = \mathsf{Cent}_{\mathcal{D}(V)}(\Bbbk[V]^G)$ denote the centraliser of $\Bbbk[V]^G$ in $\mathcal{D}(V)$. Clearly, $\mathcal{C}$ contains $\Bbbk[V]$ and $\varsigma(\mathfrak{g})$. Let $\mathcal{A}$ be the subalgebra of $\mathcal{C}$ generated by $\Bbbk[V]$ and $\varsigma(\mathfrak{g})$. Note that a vector field $\mathcal{F}$ and a polynomial $f \in \Bbbk[\mathfrak{g}]^G$ commute as differential operators if and only if $\mathcal{F}\{f\} = 0$. Therefore assertion ($\diamondsuit$) can also be interpreted as the coincidence of $\mathcal{A}$ and $\mathcal{C}$ on the level of vector fields.

Motivated by Dixmier's result [6, Theorem 2.1] and a question by Barlet, Levasseur and Stafford proved that $\mathcal{A} = \mathcal{C}$ for the adjoint representation of a semisimple Lie algebra $\mathfrak{g}$ [13]. In this section, we prove such an equality in a more general setting.

We assume below that the G-module V has the property that $\Bbbk[V]^G$ is finitely generated and the quotient field of $\Bbbk[V]^G$ equals $\Bbbk(V)^G$. The latter is equivalent to that a generic fibre of $\pi_V : V \to V /\!\!/ G$ contains a dense G-orbit.

We work with the sequence of graded $\Bbbk[V]$-modules

$$0 \to \mathsf{Ker}\,\hat{\phi} \to \mathsf{Mor}(V, \mathfrak{g}) \xrightarrow{\hat{\phi}} \mathsf{Mor}(V, V).$$

Here $\operatorname{rk}\hat{\phi} = \max_{v\in V}\dim\mathfrak{g}{\cdot}v$ [15, Prop. 1.7] and therefore $\operatorname{rk}(\operatorname{Ker}\hat{\phi}) = \min_{v\in V}\dim\mathfrak{g}_v$.

Set $m = \min_{v\in V}\dim\mathfrak{g}_v = \dim\mathfrak{g} - \dim V + \dim V/\!\!/G$. Assume that $\operatorname{Ker}\hat{\phi}$ is a free (graded) $\Bbbk[V]$-module, and let $F_1,\ldots,F_m$ be a homogeneous basis for $\operatorname{Ker}\hat{\phi}$. Write E for the $\Bbbk[V]$-module $\operatorname{Im}\hat{\phi}$. Then we obtain the exact sequence

$$(4.1) \qquad\qquad 0 \to \bigoplus_{i=1}^{m}\Bbbk[V]\,F_i \xrightarrow{\hat{\beta}} \operatorname{Mor}(V,\mathfrak{g}) \xrightarrow{\hat{\phi}} E \to 0\,.$$

Using the morphisms $F_i : V \to \mathfrak{g}$, we define a variety Y as follows:

$$Y = \{(v,\eta)\in V\times\mathfrak{g}^* \mid \langle F_i(v),\eta\rangle = 0,\ i = 1,\ldots,m\}.$$

Recall that $\operatorname{Ker}\hat{\phi}$ is a G-stable submodule of $\operatorname{Mor}(V,\mathfrak{g})$. Therefore for any $g\in G$ there exist $u_1^{(g)},\ldots,u_m^{(g)}\in\Bbbk[V]$ such that $g*F_i = \sum_j u_j^{(g)}F_j$. This readily implies that Y is G-stable.

Choose a basis $e_1,\ldots,e_n$ for $\mathfrak{g}$. Using this basis, we identify $\operatorname{Mor}(V,\mathfrak{g}) = \Bbbk[V]\otimes\mathfrak{g}$ with $\Bbbk[V]^n$. Then we can write $F_j(v) = \sum_{i=1}^{n}F_{ij}(v)e_i$, where $F_{ij}\in\Bbbk[V]$. If we regard (4.1) as a sequence

$$0 \to \Bbbk[V]^m \xrightarrow{\hat{\beta}} \Bbbk[V]^n \xrightarrow{\hat{\phi}} E \to 0\,,$$

then $\hat{\beta}$ becomes an $n\times m$-matrix with entries F_{ij}. Let $I_t(\hat{\beta})$ be the ideal of $\Bbbk[V]$ generated by $t\times t$ minors of $\hat{\beta}$. Following [7], consider the series of determinantal conditions for the presentation of E:

$$(\mathcal{F}_d) \qquad\qquad \operatorname{ht} I_t(\hat{\beta}) \geqslant m - t + 1 + d \quad \text{for } 1 \leqslant t \leqslant m.$$

The ideals $I_t(\hat{\beta})$ are independent of the presentation of E. These are *Fitting ideals* of E, see e.g. [24, 1.1]. Let $\operatorname{Sym}_{\Bbbk[V]}(E)$ denote the symmetric algebra of the $\Bbbk[V]$-module E. Then $\operatorname{Sym}_{\Bbbk[V]}(E) = \bigoplus_{n=0}^{\infty}\operatorname{Sym}_{\Bbbk[V]}(E)_n$ and each $\operatorname{Sym}_{\Bbbk[V]}(E)_n$ is a finitely generated graded $\Bbbk[V]$-module. Conditions $(\mathcal{F}_d)$ are very useful in the study of properties of the symmetric algebras of modules. Utility of these conditions in Representation and Invariant theory has been demonstrated in [15, 13, 17].

Theorem 4.1. *Suppose that* $\operatorname{Ker}\hat{\phi}$ *is a free module and condition* $(\mathcal{F}_2)$ *is satisfied by* E. *Then*

   (i) $\operatorname{Sym}_{\Bbbk[V]}(E)$ *is a factorial domain of Krull dimension* $\dim V + \dim\mathfrak{g} - m$.
  (ii) Y *is an irreducible factorial complete intersection, and* $\Bbbk[Y] = \operatorname{Sym}_{\Bbbk[V]}E$.
 (iii) $Y = \overline{\operatorname{Im}(\varkappa)}$, *where* $\varkappa : V\times V^* \to V\times\mathfrak{g}^*$ *is defined by* $\varkappa(v,\xi) = (v,\mu(v,\xi))$. *Here* $v\in V, \xi\in V^*$ *and* $\mu : V\times V^*\to\mathfrak{g}^*$ *is the moment mapping.*
  (iv) *Let* $p : Y\to V$ *be the (surjective) projection. If* $\mathfrak{I}$ *is a prime ideal of* $\Bbbk[V]$ *with* $\operatorname{ht}(\mathfrak{I})\geqslant 2$, *then* $\operatorname{ht}(\mathfrak{I}\,\Bbbk[Y])\geqslant 2$.

Proof. (i) The exact sequence (4.1) shows that E has projective dimension at most one. Therefore part (i) follows from [2, Prop. 3 & 6].

(ii) The universal property of symmetric algebras implies that $\mathsf{Sym}_{\Bbbk[V]}(E)$ is the quotient of $\mathsf{Sym}_{\Bbbk[V]}(\Bbbk[V]\otimes\mathfrak{g}) = \Bbbk[V\times\mathfrak{g}^*]$ modulo the ideal "generated by the image of $\hat{\beta}$". More precisely, each F_i determines the polynomial $\widehat{F}_i \in \Bbbk[V\times\mathfrak{g}^*]$ by the rule $\widehat{F}_i(v,\eta) = \langle F_i(v),\eta\rangle$, $\eta \in \mathfrak{g}^*$, and the ideal in question is generated by $\widehat{F}_1,\ldots,\widehat{F}_m$. Hence $\mathsf{Sym}_{\Bbbk[V]}(E) = \Bbbk[Y]$, and the other assertions follow from (i).

(iii) Clearly, $\overline{\mathrm{Im}\,(\varkappa)}$ is an irreducible subvariety of $V\times\mathfrak{g}^*$. Taking the (surjective) projection to V and looking at the dimension of the generic fibre, one finds that $\dim\overline{\mathrm{Im}\,(\varkappa)} = \dim V + \max(\dim\mathfrak{g}{\cdot}v) = \dim V + \dim\mathfrak{g} - m$. Since $F_i(v){\cdot}v = 0$ for all $v \in V$, we have

$$\widehat{F}_i(v,\mu(v,\xi)) = \langle F_i(v),\mu(v,\xi)\rangle = \langle F_i(v){\cdot}v,\xi\rangle = 0 \ .$$

Hence each $\widehat{F}_i$ vanishes on $\overline{\mathrm{Im}\,\varkappa}$ and $\overline{\mathrm{Im}\,(\varkappa)} \subset Y$. Since both varieties have the same dimension and are irreducible, they are equal.

(iv) According to [7], Remarks on pp. 664–5, this property is equivalent to condition $(\mathcal{F}_2)$. See also [24, Remark 1.3.9] $\qquad\qquad\square$

As a by-product of this theorem, we obtain the following description of $\mathsf{Sym}_{\Bbbk[V]}(E) = \Bbbk[Y]$. Consider the linear map $\mathsf{i} : \mathfrak{g} \to V^*\otimes V$ which is induced by the moment map $\mu : V\times V^* \to \mathfrak{g}^*$. (The map i is injective if and only if the representation $\mathfrak{g} \to \mathfrak{gl}(V)$ is faithful.) In this way, we obtain certain copy of $\mathfrak{g}$ sitting in $V^*\otimes V \subset \Bbbk[V\times V^*]_2$. Then $\Bbbk[Y]$ is isomorphic to the subalgebra of $\Bbbk[V\times V^*]$ generated by $\Bbbk[V]$ and $\mathsf{i}(\mathfrak{g})$.

Theorem 4.2. *Suppose that*

1. *$\Bbbk[V]^G$ is a polynomial algebra freely generated by $f_1,\ldots,f_l$;*
2. *$\mathrm{Ker}\,\hat{\phi}$ is a free $\Bbbk[V]$-module;*
3. *V has the codim–2 property and $\{(df_i)_v\}$ are linearly independent for any $v \in V_{reg}$;*
4. *condition $(\mathcal{F}_2)$ is satisfied for $E = \mathrm{Im}\,\hat{\phi}$.*

Let $\mathcal{A}$ and $\mathcal{C}$ be given the filtration induced from $\mathcal{D}(V)$. Then $\mathrm{gr}\,\mathcal{C} = \mathrm{gr}\,\mathcal{A} \simeq \mathsf{Sym}_{\Bbbk[V]}(E)$.

Proof. The proof of Levasseur and Stafford for the adjoint representation of a semisimple $\mathfrak{g}$ [13, Section 3] carries over *mutatis mutandis* to this more general situation. The following is very close to their original proof.

Lemma 4.3. *For $v \in V_{reg}$, let $\mathcal{R}$ denote the local ring of V at v. Then there exists a basis of derivations $\{\partial_i \mid i = 1,\ldots,s\}$ of $\mathrm{Der}\,\mathcal{R}$ such that $\partial_i(f_j) = \delta_{i,j}$ for all $1 \leqslant i,j \leqslant l$ and $\mathcal{R}_{\varsigma}(\mathfrak{g}) = \bigoplus_{i=l+1}^{s}\mathcal{R}\partial_i$. [Here $s = \dim V$.]*

Proof. There is a one-to-one correspondence between the bases for the $\mathcal{R}$-module $\mathrm{Der}\,\mathcal{R}$ and the $\Bbbk$-bases for the tangent space $T_v(V)$. Since $\{(df_i)_v,\ i = 1,\ldots,l\}$ are linearly independent, the annihilator of $\mathrm{span}\{(df_i)_v,\ i = 1,\ldots,l\}$ in $T_v(V)$ is $\mathfrak{g}{\cdot}v$. Choose a basis

$(e_1, \ldots, e_s)$ in V such that $\langle e_i, (df_j)_v \rangle = \delta_{i,j}, 1 \leqslant i, j \leqslant l$, and $\mathrm{span}\{e_{l+1}, \ldots, e_s\} = \mathfrak{g} \cdot v$. Then the corresponding basis for $\mathrm{Der}\,\mathcal{R}$ will work. (Cf. the proof of Lemma 3.1 in [13]). $\qquad \square$

Since $\mathrm{gr}\,\mathcal{C} \subset \mathrm{gr}\,\mathcal{D}(V) = \Bbbk[V \times V^*]$, certainly $\mathrm{gr}\,\mathcal{A} \subset \mathrm{gr}\,\mathcal{C}$ are domains. Next, $\mathrm{gr}\,\mathcal{A}$ contains the subalgebra generated by $\Bbbk[V]$ and $\mathrm{gr}\,(\varsigma(\mathfrak{g}))$. It is easily seen that $\mathrm{gr}\,(\varsigma(\mathfrak{g})) = \mathrm{i}(\mathfrak{g})$. It follows from Theorem 4.1 that, on the geometric level, we obtain the chain of dominant morphisms

$$(4.2) \qquad V \times V^* \to \mathrm{Spec}\,(\mathrm{gr}\,\mathcal{C}) \to \mathrm{Spec}\,(\mathrm{gr}\,\mathcal{A}) \to \overline{\mathrm{Im}\,\varkappa} = Y \subset V \times \mathfrak{g}^*,$$

where the resulting map $V \times V^* \to \overline{\mathrm{Im}\,\varkappa}$ is just $\varkappa$.

Lemma 4.4. *The above mapping $\varphi : \mathrm{Spec}\,(\mathrm{gr}\,\mathcal{C}) \to \overline{\mathrm{Im}\,\varkappa}$ is birational.*

Proof. We prove a more precise assertion that, for any $v \in V_{reg}$, there is the equality of local rings $(\mathrm{gr}\,\mathcal{C})_v = (\mathrm{gr}\,\mathcal{A})_v = \Bbbk[Y]_v$.

Recall that $\mathcal{R} = \Bbbk[V]_v$. For any ring $\mathfrak{R}$ containing $\Bbbk[V]$, its localisation with respect to the multiplicative subset $\{f \in \Bbbk[V] \mid f(v) \neq 0\}$ is denoted by $\mathfrak{R}_v$. Then $D(V)_v = \mathcal{R}\{\partial_1, \ldots, \partial_s\}$ is the non-commutative algebra generated by derivations constructed in Lemma 4.3 and

$$\mathcal{C}_v = \{D \in \mathcal{D}(V)_v \mid [D, f_i] = 0, \ i = 1, \ldots, l\}.$$

The last formula and Lemma 4.3 readily imply that $\mathcal{C}_v = \mathcal{A}_v = \mathcal{R}\{\partial_{l+1}, \ldots, \partial_s\}$. Let $\bar{\partial}_i$ denotes the image of ∂_i in $\mathrm{gr}\,\mathcal{D}(V)$. Lemma 4.3 also shows that

$$\mathcal{R} \cdot \mathrm{i}(\mathfrak{g}) = \bigoplus_{i=l+1}^{s} \mathcal{R}\,\bar{\partial}_i.$$

Thus, $\mathrm{gr}\,\mathcal{C}_v = \mathrm{gr}\,\mathcal{A}_v = \mathrm{gr}\,\mathcal{R}\{\partial_{l+1}, \ldots, \partial_s\} = \mathcal{R}[\bar{\partial}_{l+1}, \ldots, \bar{\partial}_s]_v = \mathcal{R}[\mathrm{i}(\mathfrak{g})]_v = \Bbbk[Y]_v.$ $\qquad \square$

Finally, we prove that $\varphi : \mathrm{Spec}\,(\mathrm{gr}\,\mathcal{C}) \to \overline{\mathrm{Im}\,\varkappa}$ is an isomorphism. We already know that φ is birational and that $\overline{\mathrm{Im}\,\varkappa}$ is normal (Theorem 4.1). By Richardson's lemma, see e.g. [3, 3.2 Lemme 1], it suffices to verify that $\mathrm{Im}\,\varphi$ contains an open subset whose complement is of codimension $\geqslant 2$. Thanks to Eq. (4.2), this reduces to the same question for $\varkappa : V \times V^* \to \overline{\mathrm{Im}\,\varkappa} \simeq Y$.

It is easily seen that if $(v, \xi) \in \overline{\mathrm{Im}\,\varkappa} = Y$ and $v \in V_{reg}$, then $(v, \xi) \in \mathrm{Im}\,\varkappa$. Invoking the projection $p : Y \to V$ shows that $p^{-1}(V_{reg})$ is on open subset lying in $\mathrm{Im}\,\varkappa$. Since V has the codim–2 property, we conclude, using Theorem 4.1(iv), that the complement of $p^{-1}(V_{reg})$ is of codimension $\geqslant 2$.

This completes the proof of Theorem 4.2. $\qquad \square$

Corollary 4.5. (i) *$\mathcal{A} = \mathcal{C}$; Moreover, $\mathcal{C}$ is an Auslander-Gorenstein, Cohen-Macaulay, Noetherian domain and a maximal order;* (ii) *the centre of $\mathcal{C}$ is $\Bbbk[V]^G$.*

(See the proof of Corollary 3.3 in [13].)

Levasseur and Stafford also prove that, in their situation, both $\mathcal{C}$ and $\operatorname{gr} \mathcal{C} = \operatorname{Sym}_{\Bbbk[V]}(E)$ are free modules over $\Bbbk[\mathfrak{g}]^G$, see [13, Corollary 3.4]. We return to this question below.

There is a particular case of Theorem 4.2, where the assumptions simplify considerably.

Proposition 4.6. *Suppose that*

1. $\Bbbk[V]^G$ *is a polynomial algebra freely generated by* $f_1, \ldots, f_l$;
2. *V has the codim–2 property and $\{(df_i)_v\}$ are linearly independent for any $v \in V_{reg}$;*
3. *there is $v \in V$ such that $\mathfrak{g}_v = 0$.*

Then $Y = \overline{\operatorname{Im} \varkappa} = V \times \mathfrak{g}^*$ and $\mathcal{A} = \mathcal{C}$.

Proof. Indeed, here $\operatorname{Ker} \hat{\phi} = 0$ and condition $(\mathcal{F}_2)$ becomes vacuous. $\square$

Verification of condition $(\mathcal{F}_2)$ is the most difficult part in possible applications of Theorem 4.2. In the rest of the section, we describe a geometrical approach to it (cf. similar approach in [17, §8]). Let us assume that the first two hypotheses of that theorem are satisfied. In particular, $\operatorname{Ker} \hat{\phi}$ is a free module (of rank m).

Using the basis morphisms $F_i : V \to \mathfrak{g}$, define the stratification of V as follows:

$$\mathfrak{X}_i = \{v \in V \mid \dim \operatorname{span}\{F_1(v), \ldots, F_m(v)\} \leqslant m - i\} .$$

Then $\mathfrak{X}_{i+1} \subset \mathfrak{X}_i$ and $\mathfrak{X}_0 = V$. As the ideal $I_t(\hat{\beta})$ defines $\mathfrak{X}_{m-t+1}$, condition $(\mathcal{F}_2)$ precisely means that $\dim \mathfrak{X}_i \leqslant \dim V - i - 2$ for any $i \geqslant 1$. In particular, $\operatorname{codim} \mathfrak{X}_1 \geqslant 3$. In case of the coadjoint representation of a 3-wonderful Lie algebra, this becomes just the *codim–3* condition on the set of non-regular points, which is used in the proof of Theorem 2.2.

Consider the quotient map $\pi_V : V \to V /\!\!/ G \simeq \mathbb{A}^l$. As usual, we say that $\pi_V^{-1}(\pi_V(0)) =: \mathcal{N} = \mathcal{N}_V$ is the *null-cone* of (the G-module) V. Set $\mathfrak{X}_i(\mathcal{N}) := \mathfrak{X}_i \cap \mathcal{N}$. Sometimes, the study of $\{\mathfrak{X}_i\}$ can be reduced to that of $\{\mathfrak{X}_i(\mathcal{N})\}$.

Recall that $\operatorname{Ker} \hat{\phi}$ is a G-stable submodule of $\operatorname{Mor}(V, \mathfrak{g})$. Therefore if $F_1, \ldots, F_m$ is a basis of $\operatorname{Ker} \hat{\phi}$, then $\{g * F_i\}_i$ is another basis for any $g \in G$. It is not clear a priori that the F_i's should be G-equivariant. Consequently, subvarieties $\mathfrak{X}_i$ are not necessarily G-stable. However, in all known examples the freeness of $\operatorname{Ker} \hat{\phi}$ does mean that there is a basis consisting of G-equivariant morphisms. (Cf. Remark 2.5 and Theorem 5.1 below). For this reason, we wish to assume that $F_i \in \operatorname{Mor}_G(V, \mathfrak{g})$.

Proposition 4.7. *Under the first two assumptions of Theorem 4.2, suppose that a generic fibre of π_V is a (closed) G-orbit, $\mathcal{N}$ contains finitely many G-orbits, and each F_i lies in $\operatorname{Mor}_G(V, \mathfrak{g})$. If*

($\clubsuit$) $\qquad\qquad\qquad\qquad \operatorname{codim}_{\mathcal{N}} \mathfrak{X}_i(\mathcal{N}) \geqslant i + 1 \text{ for any } i \geqslant 1,$

then $(\mathcal{F}_2)$ is satisfied.
An equivalent form of condition ($\clubsuit$) is that $\mathcal{N}_{reg} \subset \mathcal{X}_0(\mathcal{N})$ and, for any $v \in \mathcal{N} \setminus \mathcal{N}_{reg}$,

$$\dim(\mathsf{span}\{F_i(v) \mid i = 1, \ldots, m\}) + \mathrm{codim}_{\mathcal{N}} G{\cdot}v \geqslant m + 1.$$

Proof. The finiteness assumption guarantees us that $\dim \mathcal{N} = \dim V - \dim V /\!\!/ G$ and $\mathcal{N}_{reg} \subset V_{reg}$. Furthermore, all the fibres of π_V are of dimension $\dim V - \dim V /\!\!/ G$, see e.g. [3, Prop. 6 in p.146] (the reductivity of G is not needed in this place). By Chevalley's theorem, π_V is an open map. Consequently, it is onto. By the assumption, each $\mathcal{X}_i$ is G-stable. Let $\pi_{i,V}$ be the restriction of π_V to $\mathfrak{X}_i$. Then $\mathcal{X}_i(\mathcal{N}) = \pi_{i,V}^{-1}(\pi_{i,V}(0))$. Therefore

$$\dim \mathfrak{X}_i \leqslant \dim \mathfrak{X}_i(\mathcal{N}) + \dim \pi_V(\mathfrak{X}_i) \leqslant \dim \mathcal{N} - (i+1) + (\dim V /\!\!/ G - 1) = \dim V - (i+2).$$

Here we used the fact that $\overline{\pi_V(\mathfrak{X}_i)}$ is a proper subvariety of $V /\!\!/ G$ for $i \geqslant 1$. Indeed, $V \setminus \mathfrak{X}_1$ is a dense open subset of V and there is a dense open subset $\Xi \subset (V \setminus \mathfrak{X}_1)$ such that if $G{\cdot}v$ is a fibre of π_V for any $v \in \Xi$. The second part is an easy reformulation of condition ($\clubsuit$), which uses the finiteness for G-orbits in $\mathcal{N}$. $\qquad\qquad\square$

Recall that $v \in V_{reg}$ if and only if $\dim \mathfrak{g}_v = m$. Since $\mathrm{codim}_{\mathcal{N}} G{\cdot}v = \dim \mathfrak{g}_v - m$, yet another form of the above conditions is

(4.3) $\qquad \dim(\mathsf{span}\{F_i(v) \mid i = 1, \ldots, m\}) + \dim \mathfrak{g}_v = 2m$ for any $v \in \mathcal{N}_{reg}$,

(4.4) $\qquad \dim(\mathsf{span}\{F_i(v) \mid i = 1, \ldots, m\}) + \dim \mathfrak{g}_v \geqslant 2m+1$ for any $v \in \mathcal{N} \setminus \mathcal{N}_{reg}$.

Remark 4.8. If $\mathrm{Ker}\,\hat{\phi}$ is a free $\Bbbk[V]$-module generated by G-equivariant morphisms, then a generic stabiliser for $(G : V)$ is commutative. Indeed, the G-equivariance implies that $F_i(v)$ lies in the centre of $\mathfrak{g}_v$ for any $v \in V$. On the other hand, if v is generic, then $F_1(v), \ldots, F_m(v)$ form a basis for $\mathfrak{g}_v$. (Cf. Remark 3.2 in [18].)

The above inequality ($\clubsuit$) is crucial for establishing $(\mathcal{F}_2)$ in applications. Furthermore, it essentially implies that $\mathcal{C}$ to be a free $\Bbbk[V]^G$-module.

Theorem 4.9. *Suppose that*

1. *$\Bbbk[V]^G$ is a polynomial algebra freely generated by $f_1, \ldots, f_l$;*
2. *V has the codim–2 property and $\{(df_i)_v\}$ are linearly independent for any $v \in V_{reg}$;*
3. *$\mathrm{Ker}\,\hat{\phi}$ is a free $\Bbbk[V]$-module generated G-equivariant morphisms $F_1, \ldots, F_m$;*
4. *a generic fibre of π_V is a (closed) G-orbit;*
5. *$\mathcal{N}$ contains finitely many G-orbits;*
6. *$\mathrm{codim}_{\mathcal{N}} \mathcal{X}_i(\mathcal{N}) \geqslant i + 1$ for any $i \geqslant 1$.*

Then $\mathrm{gr}\,\mathcal{A} = \mathrm{gr}\,\mathcal{C} = \mathsf{Sym}_{\Bbbk[V]}(\mathrm{Im}\,\hat{\phi})$, $\mathcal{A} = \mathcal{C}$, and both $\mathcal{C}$ and $\mathrm{gr}\,\mathcal{C}$ are free (left or right) $\Bbbk[V]^G$-modules.

Proof. As the hypotheses imply condition $(\mathcal{F}_2)$ for $E = \operatorname{Im}\hat\phi$, only the last assertion requires a proof.

Recall that $\Bbbk[Y] = \operatorname{gr}\mathcal{C}$ and $Y = \overline{\operatorname{Im}(\varkappa)}$ is a complete intersection of codimension m in $V \times \mathfrak{g}^*$. Consider the map $\nu : Y \to V \to V/\!/G \simeq \mathbb{A}^l$, the composition of the projection and the quotient morphism π_V. Its fibre over the origin is

$$\mathcal{Z} := \{(v,\xi) \mid v \in \mathcal{N} \ \& \ \langle F_i(v), \xi \rangle = 0, \ i = 1, \ldots, m\}.$$

We wish to prove that $\mathcal{Z}$ is a variety of pure dimension $\dim Y - l$. Since $\mathcal{Z}$ is a fibre of a dominant map $Y \to \mathbb{A}^l$, we have $\dim \mathcal{Z} \geqslant \dim Y - l$. On the other hand, consider the projection $p : \mathcal{Z} \to \mathcal{N}$. It follows from hypothesis 6 that

$$\dim p^{-1}(\mathcal{X}_i(\mathcal{N}) \setminus \mathcal{X}_{i+1}(\mathcal{N})) \leqslant \dim \mathcal{N} - i - 1 + \dim \mathfrak{g} - m + i = \dim Y - l - 1$$

for any $i \geqslant 1$. Hence $\mathcal{Z} = \overline{p^{-1}(\mathcal{N} \setminus \mathcal{X}_1(\mathcal{N}))}$ is of pure dimension $\dim Y - l$. [Furthermore, it is not hard to prove that p provides a one-to-one correspondence between the irreducible components of $\mathcal{Z}$ and $\mathcal{N}$.] Consequently, ν is equidimensional. Since Y is Cohen-Macaulay, the ν is also flat. This shows that each $(\operatorname{gr}\mathcal{C})_n$ is a flat graded finitely generated module over the polynomial ring $\Bbbk[V]^G$, hence a free module. Thus, $\operatorname{gr}\mathcal{C}$ is a free $\Bbbk[V]^G$-module.

The assertion on $\mathcal{C}$ can be proved exactly as in Corollary 3.4 in [13]. $\qquad\square$

Remark 4.10. Conditions $(\mathcal{F}_d)$ can also be considered for $\operatorname{Im}\hat\psi$ or $\operatorname{Im}\hat\tau$ whenever $\operatorname{Ker}\hat\psi$ is a free $\Bbbk[V]$-module or $\operatorname{Ker}\hat\tau$ is a free $\Bbbk[\mathfrak{g}]$-module. If $(\mathcal{F}_2)$ is satisfied, then one obtains a similar description for the image of $\varkappa_\psi : V \times \mathfrak{g} \to V \times V$, $(v,x) \mapsto (v, x{\cdot}v)$ or $\varkappa_\tau : \mathfrak{g} \times V \to \mathfrak{g} \times V$, $(x,v) \mapsto (x, x{\cdot}v)$, see Theorems 8.8 and 8.11 in [17]. However such descriptions seem to have no non-commutative counterparts.

5. APPLICATIONS: ISOTROPY REPRESENTATIONS OF SYMMETRIC PAIRS AND BEYOND

In this section, G is a connected *semisimple* algebraic group. If $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $\mathbb{Z}_2$-grading of $\mathfrak{g}$, then $(\mathfrak{g}, \mathfrak{g}_0)$ is said to be a *symmetric pair*. Let G_0 be the connected subgroup of G with Lie algebra $\mathfrak{g}_0$. Our goal is to describe a class of $\mathbb{Z}_2$-gradings that lead to isotropy representations $(G_0 : \mathfrak{g}_1)$ satisfying the assumptions of Theorems 4.2 and 4.9.

Recall necessary results on the representation $(G_0 : \mathfrak{g}_1)$. The standard reference for this is [10]. Let $\mathcal{N}$ denote the set of nilpotent elements of $\mathfrak{g}$.

⟨1⟩ Any $v \in \mathfrak{g}_1$ admits a unique decomposition $v = v_s + v_n$, where $v_s \in \mathfrak{g}_1$ is semisimple and $v_n \in \mathcal{N} \cap \mathfrak{g}_1$; $v = v_s$ if and only if $G_0{\cdot}v$ is closed; $v = v_n$ if and only if the closure of $G_0{\cdot}v$ contains the origin. For any $v \in \mathfrak{g}_1$, there is the induced $\mathbb{Z}_2$-grading of the centraliser $\mathfrak{g}_v = \mathfrak{g}_{0,v} \oplus \mathfrak{g}_{1,v}$, and $\dim \mathfrak{g}_0 - \dim \mathfrak{g}_1 = \dim \mathfrak{g}_{0,v} - \dim \mathfrak{g}_{1,v}$.

⟨2⟩ Let $\mathfrak{c} \subset \mathfrak{g}_1$ be a maximal subspace consisting of pairwise commuting semisimple elements. All such subspaces are G_0-conjugate and $G_0 \cdot \mathfrak{c}$ is dense in $\mathfrak{g}_1$; $\dim \mathfrak{c}$ is called the *rank* of the $\mathbb{Z}_2$-grading or pair $(\mathfrak{g}, \mathfrak{g}_0)$, denoted $\mathrm{rk}\,(\mathfrak{g}, \mathfrak{g}_0)$. If $v \in \mathfrak{c} \cap (\mathfrak{g}_1)_{reg}$, then $\mathfrak{g}_{1,v} = \mathfrak{c}$ and $\mathfrak{g}_{0,v}$ is a generic stabiliser for the G_0-module $\mathfrak{g}_1$.

⟨3⟩ The algebra $\mathbb{k}[\mathfrak{g}_1]^{G_0}$ is polynomial and $\dim \mathfrak{g}_1 /\!/ G_0 = \mathrm{rk}\,(\mathfrak{g}, \mathfrak{g}_0)$. The quotient map $\pi_{\mathfrak{g}_1} : \mathfrak{g}_1 \to \mathfrak{g}_1 /\!/ G_0$ is equidimensional. The generic fibre is the orbit of a G_0-regular semisimple element. Each fibre of $\pi_{\mathfrak{g}_1}$ contains finitely many G_0-orbits and each closed G_0-orbit in $\mathfrak{g}_1$ meets $\mathfrak{c}$. The null-cone in $\mathfrak{g}_1$ equals $\mathcal{N} \cap \mathfrak{g}_1$.

⟨4⟩ If $v \in (\mathfrak{g}_1)_{reg}$ and $f_1, \ldots, f_{\dim \mathfrak{c}} \in \mathbb{k}[\mathfrak{g}_1]^{G_0}$ are basis invariants, then the $\{(df_i)_v\}_i$ are linearly independent.

A $\mathbb{Z}_2$-grading (or a symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$) is said to be $\mathcal{N}$-*regular* if $\mathfrak{g}_1$ contains a regular nilpotent element of $\mathfrak{g}$. By a result of Antonyan [1], a $\mathbb{Z}_2$-grading is $\mathcal{N}$-regular if and only if $\mathfrak{g}_1$ contains a regular semisimple element. Then $\dim \mathfrak{g}_0 - \dim \mathfrak{g}_1 = \mathrm{rk}\,\mathfrak{g} - 2\dim \mathfrak{c}$. Now we are interested in properties of

$$(5.1) \qquad 0 \to \mathrm{Ker}\,\hat{\phi} \to \mathrm{Mor}(\mathfrak{g}_1, \mathfrak{g}_0) \xrightarrow{\hat{\phi}} \mathrm{Mor}(\mathfrak{g}_1, \mathfrak{g}_1)$$

for $\mathcal{N}$-regular $\mathbb{Z}_2$-gradings. Item ⟨2⟩ above shows that a generic stabiliser for $(G_0 : \mathfrak{g}_1)$ is commutative if and only if $\mathfrak{c}$ contains a regular semisimple element. Therefore a $\mathbb{Z}_2$-grading is $\mathcal{N}$-regular if and only if a generic stabiliser for $(G_0 : \mathfrak{g}_1)$ is commutative. In view of Remark 4.8, these are the only $\mathbb{Z}_2$-gradings, where one could expect that $\mathrm{Ker}\,\hat{\phi}$ is generated by G_0-equivariant morphisms. The following is [18, Theorem 5.8].

Theorem 5.1. *Suppose the $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is $\mathcal{N}$-regular. Then $\mathrm{Ker}\,\hat{\phi}$ is a free $\mathbb{k}[\mathfrak{g}_1]$-module generated by G_0-equivariant morphisms. In this case $\mathrm{rk}\,(\mathrm{Ker}\,\hat{\phi}) = \mathrm{rk}\,\mathfrak{g} - \dim \mathfrak{c}$.*

Recall the construction of such a basis for $\mathrm{Ker}\,\hat{\phi}$. By [16, Theorem 4.7], $Z = \overline{G \cdot \mathfrak{g}_1}$ is a normal complete intersection in $\mathfrak{g}$, $\mathrm{codim}\,Z = \mathrm{rk}\,\mathfrak{g} - \dim \mathfrak{c}$, and the ideal of Z in $\mathbb{k}[\mathfrak{g}]$ is generated by certain homogeneous basis invariants in $\mathbb{k}[\mathfrak{g}]^G$. Let $f_1, \ldots, f_m$ be such basis invariants ($m = \mathrm{rk}\,\mathfrak{g} - \dim \mathfrak{c}$). As in Section 4, any $F_i \in \mathrm{Ker}\,\hat{\phi}$ determines the polynomial $\widehat{F}_i \in \mathbb{k}[\mathfrak{g}_1 \times \mathfrak{g}_0^*] = \mathbb{k}[\mathfrak{g}_1 \times \mathfrak{g}_0] \simeq \mathbb{k}[\mathfrak{g}]$ and vice versa. In this case, $\widehat{F}_i$ is defined to be the bi-homogeneous component of f_i of degree $(1, \deg f_i - 1)$. (Here "1" is the degree with respect to $\mathfrak{g}_0$.) Since $\widehat{F}_i$ is clearly G_0-invariant, F_i is G_0-equivariant. From this description, it follows that $(df_i)_v = F_i(v)$ if $v \in \mathfrak{g}_1$.

For $\mathfrak{g}$ simple, the list of $\mathcal{N}$-regular symmetric pairs consists of symmetric pairs of maximal rank (when $\dim \mathfrak{c} = \mathrm{rk}\,\mathfrak{g}$ and hence $\mathrm{Ker}\,\hat{\phi} = 0$) and the following 4 cases:

1	$(\mathfrak{sl}_{2n}, \mathfrak{sl}_n \dotplus \mathfrak{sl}_n \dotplus \mathfrak{t}_1)$	$m = \mathrm{rk}\,(\mathrm{Ker}\,\hat{\phi}) = n-1$;
2	$(\mathfrak{sl}_{2n+1}, \mathfrak{sl}_n \dotplus \mathfrak{sl}_{n+1} \dotplus \mathfrak{t}_1)$	$m = \mathrm{rk}\,(\mathrm{Ker}\,\hat{\phi}) = n$;
3	$(\mathfrak{so}_{2n}, \mathfrak{so}_{n-1} \dotplus \mathfrak{so}_{n+1}), n \geqslant 4$	$m = \mathrm{rk}\,(\mathrm{Ker}\,\hat{\phi}) = 1$;
4	$(\mathbf{E}_6, \mathbf{A}_5 \times \mathbf{A}_1)$	$m = \mathrm{rk}\,(\mathrm{Ker}\,\hat{\phi}) = 2$.

Theorem 5.2. *Suppose the $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is $\mathcal{N}$-regular, and $(\mathfrak{g}, \mathfrak{g}_0) \neq (\mathfrak{sl}_{2n+1}, \mathfrak{sl}_n \dotplus \mathfrak{sl}_{n+1} \dotplus \mathfrak{t}_1)$. Then inequality ($\clubsuit$) holds for $\mathcal{N}_{\mathfrak{g}_1} = \mathcal{N} \cap \mathfrak{g}_1$ and therefore $(\mathcal{F}_2)$ is satisfied for $\operatorname{Im} \hat{\phi}$ in Eq. (5.1).*

Proof. In the maximal rank case, $\operatorname{Ker} \hat{\phi} = 0$ and therefore condition $(\mathcal{F}_2)$ is vacuous. In the remaining four cases, we have to resort to explicit calculations. By results of [10], Proposition 4.7 applies in this situation. Hence our goal is to verify whether Eq. (4.3) and (4.4) are satisfied for the elements of $\mathcal{N} \cap \mathfrak{g}_1$.

Let $F_1, \ldots, F_m$ be a basis for $\operatorname{Ker} \hat{\phi}$. The above (classification-free) construction of the F_i's implies that $\dim(\operatorname{span}\{F_1(v), \ldots, F_m(v)\}) = m$ for any $v \in (\mathfrak{g}_1)_{reg}$ (see the beginning of Section 5 in [18]). Therefore Eq. (4.3) is true in all four cases. It remains to handle Eq. (4.4).

In the $\mathcal{N}$-regular case, each nilpotent G-orbit meets $\mathfrak{g}_1$ [1]. Therefore we can argue in terms of nilpotent G-orbits in $\mathfrak{g}$. Consider all the cases in turn.

For the first two cases, the explicit form of the F_i's is pointed out in [18, Example 5.6]. Namely, regarding elements $v \in \mathfrak{g}_1$ as matrices (of order $2n$ or $2n + 1$), we set $F_i(v) = v^{2i}$, the usual matrix power with $1 \leqslant i \leqslant m$.

Remark. Strictly speaking, this formula for F_i is only valid if the big Lie algebra is $\mathfrak{gl}_N$. For $\mathfrak{sl}_N$, one have to add a correcting term in order to ensure zero trace: $F_i(v) = v^{2i} - (\operatorname{tr}(v^{2i})/N)I$. However, the correcting term vanishes if v is nilpotent, which is the case below.

No. $\boxed{1}$ Let $(\eta_1, \eta_2, \ldots)$ be the partition of $2n$ corresponding to $v \in \mathcal{N} \cap \mathfrak{g}_1$. Then $v^{2i} \neq 0$ if and only if $2i \leqslant \eta_1 - 1$, and the nonzero terms are easily seen to be linearly independent. Therefore $\dim(\operatorname{span}\{F_1(v), \ldots, F_m(v)\}) = \lfloor \frac{\eta_1 - 1}{2} \rfloor$. Write $(\hat{\eta}_1, \hat{\eta}_2, \ldots, \hat{\eta}_s)$ for the dual partition. Then $s = \eta_1$. The term $\mathfrak{g}_v$ occurring in Eq. (4.4) now becomes $\mathfrak{g}_{0,v}$. The general equality $\dim G{\cdot}v = 2 \dim G_0{\cdot}v$ means in this case that $\dim \mathfrak{g}_v = 2 \dim \mathfrak{g}_{0,v} + 1$. Because $\dim \mathfrak{g}_v = \dim(\mathfrak{sl}_{2n})_v = \sum_{i=1}^{s} \hat{\eta}_i^2 - 1$, the required inequality looks as follows:

$$\frac{1}{2} \sum_{i=1}^{s} \hat{\eta}_i^2 + \left\lfloor \frac{\eta_1 - 1}{2} \right\rfloor - 2n \geqslant 0,$$

if $v \notin (\mathcal{N} \cap \mathfrak{g}_1)_{reg}$, i.e., if $\hat{\eta}_1 \geqslant 2$. Since $\sum_{i=1}^{s} \hat{\eta}_i = 2n$, it is readily transformed into

$$\frac{1}{2} \sum_{i=1}^{s} (\hat{\eta}_i - 1)^2 + \left(\left\lfloor \frac{s-1}{2} \right\rfloor - \frac{s}{2} \right) \geqslant 0 ,$$

which is true if $\hat{\eta}_1 \geqslant 2$. Indeed, if v is subregular, then $\hat{\eta}_1 = 2$, $\hat{\eta}_j = 1$ for $j \geqslant 2$, and $s = 2n-1$. Hence the LHS is zero. For all other non-regular SL_{2n}-orbits the LHS is positive.

No. $\boxed{2}$ In this case, the numerical data are slightly different: $\sum_{i=1}^{s} \hat{\eta}_i = 2n + 1$, $m = n$, and $\dim \mathfrak{g}_v = 2 \dim \mathfrak{g}_{0,v}$. The required inequality is

$$\frac{1}{2} \sum_{i=1}^{s} (\hat{\eta}_i - 1)^2 + \left(\left\lfloor \frac{s-1}{2} \right\rfloor - \frac{s+1}{2} \right) \geqslant 0 \, ,$$

if $v \notin (\mathcal{N} \cap \mathfrak{g}_1)_{reg}$. It fails to hold only if v is subregular, i.e., $\hat{\eta}_1 = 2$, $\hat{\eta}_j = 1$ for $j \geqslant 2$, and $s = 2n$. This means that $\operatorname{codim}_{\mathcal{N}} \mathcal{X}_1(\mathcal{N}) = 1$ and ($\clubsuit$) in Proposition 4.7 is not satisfied. Using this, one can prove that condition $(\mathcal{F}_2)$ is not satisfied for $\operatorname{Im} \hat{\phi}$ here.

No. $\boxed{3}$ Since F_1 is the only basis morphism, the validity of Eq. (4.4) reduces to the assertion that $F_1(v) \neq 0$ whenever $G_0 \cdot v$ is an orbit of codimension 1 in $\mathcal{N} \cap \mathfrak{g}_1$. The map F_1 arises from the pfaffian, Pf, a skew-symmetric matrix of order $2n$, and, as explained above, $F_1(u) = d(\mathsf{Pf})_u$. If $u \in \mathfrak{so}_{2n}$ is nilpotent, then $d(\mathsf{Pf})_u = 0$ if and only if u has at least three Jordan blocks [22, Lemma 4.4.1]. However $G \cdot v \subset \mathcal{N}$ is the subregular nilpotent orbit and the corresponding partition is $(2n - 1, 3)$.

No. $\boxed{4}$ Here $m = 2$ and there are two basis morphisms in $\operatorname{Ker} \hat{\phi}$. These two are associated with basis invariants of $G = \mathbf{E}_6$ of degree 5 and 9. Therefore their degrees are equal to 4 and 8. Call them $F^{(4)}$ and $F^{(8)}$, respectively. Here the validity of Eq. (4.4) reduces to the assertions that

$$\begin{cases} \text{if } \operatorname{codim}_{\mathcal{N} \cap \mathfrak{g}_1} G_0 \cdot v = 1, \text{ then } F^{(4)}(v) \neq 0 \text{ and } F^{(8)}(v) \neq 0, \\ \text{if } \operatorname{codim}_{\mathcal{N} \cap \mathfrak{g}_1} G_0 \cdot v = 2, \text{ then } F^{(4)}(v) \neq 0 \text{ or } F^{(8)}(v) \neq 0. \end{cases}$$

In the first case, $G \cdot v$ is the subregular nilpotent orbit, usually denoted $\mathbf{E}_6(a_1)$. In the second case, $G \cdot v$ is the unique orbit of codimension 4 in $\mathcal{N}$, denoted $\mathbf{D}_5$. The computations we need have been performed by Richardson, see [22, Appendix]. He computed the "exponents" for all but one nilpotent orbits in the exceptional Lie algebras. In particular, for the G-orbit of type $\mathbf{E}_6(a_1)$ (resp. $\mathbf{D}_5$), the exponents include 4 and 8 (resp. include 4). This is exactly what we need. $\qquad \square$

It follows from the previous exposition that if $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is an $\mathcal{N}$-regular grading, then, modulo one exception, all the hypotheses of Theorem 4.9 are satisfied for the G_0-module $\mathfrak{g}_1$. Indeed, by above-mentioned results of Kostant and Rallis [10], the hypotheses 1, 2, 4, and 5 hold for all $\mathbb{Z}_2$-gradings. The third (resp. sixth) assumption is verified in Theorem 5.1 (resp. Theorem 5.2).

Thus, applying results of Section 4 to our situation, we obtain

Theorem 5.3. *Suppose that $\mathfrak{g}$ is simple, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is an $\mathcal{N}$-regular $\mathbb{Z}_2$-grading, and $(\mathfrak{g}, \mathfrak{g}_0) \neq (\mathfrak{sl}_{2n+1}, \mathfrak{sl}_n \dotplus \mathfrak{sl}_{n+1} \dotplus \mathfrak{t}_1)$. Set $m = \operatorname{rk} \mathfrak{g} - \dim \mathfrak{c}$. Then*

(i) $\operatorname{Sym}_{\Bbbk[\mathfrak{g}_1]}(\operatorname{Im} \hat{\phi})$ *is a factorial domain of Krull dimension* $\dim \mathfrak{g}_1 + \dim \mathfrak{g}_0 - m$.

(ii) $\mathrm{Spec}\,(\mathsf{Sym}_{\Bbbk[\mathfrak{g}_1]}(\mathrm{Im}\,\hat{\phi})) \simeq \overline{\mathrm{Im}\,(\varkappa)}$, *where* $\varkappa : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_1 \times \mathfrak{g}_0$ *is defined by* $(v, v') \mapsto$ $(v, [v, v'])$.

(iii) $\overline{\mathrm{Im}\,(\varkappa)}$ *is an irreducible factorial complete intersection and its ideal is generated by* $\widehat{F}_i$, $i = 1, \ldots, m$.

(iv) $\mathcal{A} = \mathcal{C}$, *i.e.,* $\mathcal{C} = \mathsf{Cent}_{\mathcal{D}(\mathfrak{g}_1)}(\Bbbk[\mathfrak{g}_1]^{G_0})$ *is the algebra generated by* $\Bbbk[\mathfrak{g}_1]$ *and* $\varsigma(\mathfrak{g}_0)$.

(v) *Both* $\mathcal{C}$ *and* $\mathrm{gr}\,\mathcal{C} = \mathsf{Sym}_{\Bbbk[\mathfrak{g}_1]}(\mathrm{Im}\,\hat{\phi})$ *are free (left or right)* $\Bbbk[\mathfrak{g}_1]^{G_0}$-*modules.*

Remark 5.4. For the symmetric pairs of maximal rank, the equality $\mathcal{A} = \mathcal{C}$ is proved in an unpublished manuscript of Levasseur [11, Theorem 4.4]. In our exposition, this assertion also appears as a special case of Proposition 4.6.

Remark 5.5. As $(\mathcal{F}_2)$ is not satisfied for $\mathrm{Im}\,\hat{\phi}$ in case of $(\mathfrak{sl}_{2n+1}, \mathfrak{sl}_n \dotplus \mathfrak{sl}_{n+1} \dotplus \mathfrak{t}_1)$, one might expect that some assertions of Theorem 5.3 are wrong for that symmetric pair. However, condition $(\mathcal{F}_1)$ still holds, and this is sufficient for proving that $Y = \overline{\mathrm{Im}\,(\varkappa)}$ and it is a complete intersection whose ideal is generated by $\widehat{F}_1, \ldots, \widehat{F}_m$. It seems to be hard to check directly what is happening with assertion (iv). We are only able to prove that $\overline{\mathrm{Im}\,(\varkappa)}$ is <u>not</u> factorial for $(\mathfrak{sl}_3, \mathfrak{gl}_2)$. For, here $\overline{\mathrm{Im}\,(\varkappa)}$ is a hypersurface in the 8-dimensional space $\mathfrak{g}_1 \times \mathfrak{g}_0$, and the defining relation can be written up.

As Theorems 4.2 and 4.9 have rather general formulations (the group G even is not supposed to be reductive!), it is instructive to have natural illustrations to it, which lie outside the realm of the isotropy representations of symmetric pairs. In view of Proposition 4.6, many representations with trivial generic stabiliser will work. So, we concentrate on examples with non-trivial stabiliser, i.e., with $\mathrm{Ker}\,\hat{\phi} \neq 0$.

Example 5.6. Take $G = SL_6 \times SL_3$ and $V = \mathsf{R}(\varpi_2) \otimes \mathsf{R}(\varpi_1)$, where ϖ_i is the i-th fundamental weight. This representation is associated with an automorphism of order 3 of $\mathbf{E}_7$, and Vinberg's theory of θ-groups [25], which is an extension of the Kostant-Rallis theory, provides a lot of information about it. In particular, $\Bbbk[V]^G$ is polynomial (with three generators) and $\mathcal{N}_V$ contains finitely many G-orbits. Here $m = 1$ and Proposition 4.7 is applicable. The situation here resembles very much that for $\mathcal{N}$-regular $\mathbb{Z}_2$-gradings. The basis covariant $F : V \to \mathfrak{g}^*$ in $\mathrm{Ker}\,\hat{\phi}$ is associated with the basis $\mathbf{E}_7$-invariant f of degree 10. Therefore $\deg F = 9$ and $F(v) = (df)_v$ for $v \in V$. Since $m = 1$, it suffices to verify that $\dim \mathcal{N}_V - \dim \mathcal{X}_1(\mathcal{N}_V) \geqslant 2$. In other words, if $\mathcal{O}$ is a G-orbit of codimension 1 in $\mathcal{N}_V$, then we need $F|_{\mathcal{O}} \neq 0$. An explicit classification of G-orbits in $\mathcal{N}_V$ [4, §4 Table 8] shows that orbits of codimension 1 lie inside of nilpotent $\mathbf{E}_7$-orbits denoted by $\mathbf{E}_7(a_1)$ and $\mathbf{E}_7(a_2)$. Finally, using again Richardson's calculations [22, Appendix], we obtain the required non-vanishing assertion.

We omit most details for this example, since we are going to consider applications of our theory to θ-groups in a forthcoming article.

Example 5.7. We describe non-reductive Lie algebras whose coadjoint representation sat-
isfies the hypotheses of Theorem 4.2. This yields an incarnation of the "$\mathcal{A} = \mathcal{C}$ phe-
nomenon" in the non-reductive case. We detect such examples among 3-wonderful alge-
bras. By a previous discussion (Remarks 2.1 and 2.5), hypotheses 1–3 are always satisfied
for them. Thus, it remains only to have condition ($\mathcal{F}_2$) for $\mathrm{Im}\,\hat\phi$. Our examples exploit
the semi-direct product construction (see Example 2.6). We start with $\mathfrak{s} = \mathfrak{sl}_2$ and set
$\mathfrak{q} = \mathfrak{sl}_2 \ltimes \mathfrak{sl}_2$. Then $\mathfrak{q}$ is a quadratic 3-wonderful algebra and $m = \mathrm{ind}\,\mathfrak{q} = 2$. There are two
basis Q-equivariant morphisms $F_i : \mathfrak{q}^* \to \mathfrak{q}$ in $\mathrm{Ker}\,\hat\phi$. Identifying $\mathfrak{q}^*$ and $\mathfrak{q}$, we may regard
F_i as elements of $\mathrm{Mor}_Q(\mathfrak{q}, \mathfrak{q})$. Representing elements of $\mathfrak{q}$ as pairs (x, y), where $x, y \in \mathfrak{sl}_2$,
we obtain the following explicit formulae: $F_1(x, y) = (x, y)$ and $F_2(x, y) = (0, x)$. Then
$\mathfrak{X}_1 = \{(0, y) \mid y \in \mathfrak{sl}_2\}$ and $\mathfrak{X}_2 = \{(0, 0)\}$. Hence condition ($\mathcal{F}_2$) is satisfied. We have also
checked ($\mathcal{F}_2$) for $\mathfrak{sl}_3 \ltimes \mathfrak{sl}_3$, $\mathfrak{sp}_4 \ltimes \mathfrak{sp}_4$ and $\mathbf{G}_2 \ltimes \mathbf{G}_2$.

Hopefully, this could be true if $\mathfrak{s}$ is any simple Lie algebra, but we unable to prove it.

6. SOME SPECULATIONS

There are two different generalisations of Dixmier's result on adjoint vector fields in the
context of the adjoint representation of semisimple Lie algebras.

One possibility is provided by the Levasseur-Stafford description of $\mathrm{Cent}_{\mathcal{D}(\mathfrak{g})}(\Bbbk[\mathfrak{g}]^G)$ [13,
Theorem 1.1], see discussion in Section 4. On the other hand, the formulation given in
Remark 2.4 suggests the following question, which was raised by Dixmier himself [6,
1.2].

Question. Suppose $D \in \mathcal{D}(\mathfrak{g})$ and $D(f) = 0$ for all $f \in \Bbbk[\mathfrak{g}]^G$. Is it true that $D \in \mathcal{D}(\mathfrak{g}){\cdot}\varsigma(\mathfrak{g})$?

The affirmative answer is obtained by Levasseur and Stafford [12]. They proved that

$$\mathcal{K} := \{D \in \mathcal{D}(\mathfrak{g}) \mid D(f) = 0 \ \forall f \in \Bbbk[\mathfrak{g}]^G\}$$

is the left ideal of $\mathcal{D}(\mathfrak{g})$ generated by $\varsigma(\mathfrak{g})$. Then a similar result was obtained for the
isotropy representation of some symmetric pairs [14]. To state that result, we need some
preparations. Let Σ be the restricted root system of $(\mathfrak{g}, \mathfrak{g}_0)$. The following condition on Σ
was considered by Sekiguchi [23]:

($\heartsuit$) $\dim \mathfrak{g}_\alpha + \dim \mathfrak{g}_{2\alpha} \leqslant 2$ for any $\alpha \in \Sigma$.

Sekiguchi also obtained the list of corresponding symmetric pairs. The following is [14,
Theorem C]:

*Suppose that $\Sigma(\mathfrak{g}, \mathfrak{g}_0)$ satisfies ($\heartsuit$). Then $\mathcal{K}(\mathfrak{g}_1) := \{D \in \mathcal{D}(\mathfrak{g}_1) \mid D(f) = 0 \ \forall f \in \Bbbk[\mathfrak{g}_1]^{G_0}\}$ is
the left ideal generated by $\varsigma(\mathfrak{g}_0)$.*

Furthermore, it is proved in [14, Section 6] that $\mathcal{K}(\mathfrak{g}_1) = \mathcal{D}(\mathfrak{g}_1)\varsigma(\mathfrak{g}_0)$ for $(\mathfrak{so}_{n+1}, \mathfrak{so}_n)$, while the inclusion $\mathcal{D}(\mathfrak{g}_1)\varsigma(\mathfrak{g}_0) \subset \mathcal{K}(\mathfrak{g}_1)$ is strict for $(\mathfrak{sl}_3, \mathfrak{gl}_2)$. It is curious that according to Sekiguchi's classification, $(\heartsuit)$ is satisfied precisely if $(\mathfrak{g}, \mathfrak{g}_0)$ is $\mathcal{N}$-regular except for $(\mathfrak{sl}_{2n+1}, \mathfrak{sl}_n \dot{+} \mathfrak{sl}_{n+1} \dot{+} \mathfrak{t}_1)$.

This raises the following questions for representations of connected (reductive?) groups. There are two properties of representations $(G : V)$:

1) The algebra $\mathsf{Cent}_{\mathcal{D}(V)}(\Bbbk[V]^G)$ is generated by $\Bbbk[V]$ and $\varsigma_V(\mathfrak{g})$;

2) The ideal $\{D \in \mathcal{D}(V) \mid D(f) = 0 \ \forall f \in \Bbbk[V]^G\}$ is generated by $\varsigma_V(\mathfrak{g})$.

Is it true that one of them implies another (under appropriate constraints)? At least, is there a relationship in case of isotropy representations of symmetric pairs?

Remark. The only "bad" $\mathcal{N}$-regular symmetric pair $(\mathfrak{sl}_{2n+1}, \mathfrak{sl}_n \dot{+} \mathfrak{sl}_{n+1} \dot{+} \mathfrak{t}_1)$ is also distinguished by a bad behaviour of the commuting variety. Recall that the commuting variety is $\mathfrak{E}(\mathfrak{g}_1) = \{(x, y) \in \mathfrak{g}_1 \times \mathfrak{g}_1 \mid [x, y] = 0\}$, and it is irreducible for all $\mathcal{N}$-regular pairs but that one. In the maximal rank case, the irreducibility is proved in [15]. The four remaining cases (see the list in Section 5) are considered in [19]. This is of certain interest because there is a relationship between the irreducibility of $\mathfrak{E}(\mathfrak{g}_1)$ and properties of the ideal $\mathcal{K}(\mathfrak{g}_1)$, see [14, Prop. 4.6].

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